Concircular curvature tensor in Kenmotsu manifold

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Abstract
In this paper we study Concircular curvature tensor in Kenmotsu manifolds. We consider curvature tensor, scalar curvature in a Kenmotsu manifold admitting concircular curvature tensor and prove the condition for concircular curvature tensor to be shrinking, steady and expanding.

Keywords: Curvature tensor, scalar curvature, Ricci tensor, Einstein, Kenmotsu manifold

Introduction
The Concircular curvature Ĉ on Kenmostu manifolds of dimensional n is defined by

\[
\tilde{C}(X, Y)\zeta = R(X, Y)\zeta - r \frac{n}{n-1} [g(Y, \zeta)X - g(X, \zeta)Y]
\]

for any vector fields X, Y, Z where R is the curvature tensor and r is the scalar curvature.

In an n-dimensional Kenmotsu manifold, we have

\[
\eta(R(X, Y)Z) = g(X, Z) \eta(Y) - g(Y, Z) \eta(X),
\]

(1.2)

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X,
\]

(1.3)

\[
R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,
\]

(1.4)

\[
R(\xi, X)\xi = X - \eta(X)\xi,
\]

(1.5)

where R is the Riemannian curvature tensor.

Taking Z = \xi in equation (1.1) and using equation (1.3), we get

\[
\tilde{C}(X, Y)\xi = R(X, Y)\xi - \frac{r}{n(n-1)} [g(Y, \xi)X - g(X, \xi)Y]
\]

⇒ \tilde{C}(X, Y)\xi = \left[1 + \frac{r}{n(n-1)}\right] [\eta(X)Y - \eta(Y)X].

(1.6)

Taking inner product of equation (1.1) with \xi, we get

\[
\eta(\tilde{C}(X, Y)Z) = \eta(R(X, Y)Z) - \frac{r}{n(n-1)} [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\]

(1.7)

Using equation (1.2) in above equation, we get

\[
\eta(\tilde{C}(X, Y)Z) = \left[1 + \frac{r}{n(n-1)}\right] [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]
\]

(1.8)

2.2 Preliminaries
An n-dimensional differential manifold M is said to be an almost contact metric manifold \[^3\]. If it admits an almost contact metric structure (\phi, \xi, \eta, g) consisting of a tensor field \phi of type
(1, 1) a vector field \( \xi \), a 1 - form \( \eta \), and a Riemannian metric \( g \) compatible with \((\phi, \xi, \eta, g)\) satisfying.

\[
\phi \xi = -1 + \mathcal{L}_\xi \phi, \eta (\xi) = 1, \tag{2.1}
\]

\[
\eta \ast \phi = 0, \phi \xi = 0 \tag{2.2}
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta (X) \eta (Y), \ g(\xi, \xi) = \eta (X). \tag{2.3}
\]

for all vector fields \( X, Y \) on \( M \).

An almost contact metric manifold \( (\phi, \xi, \eta, g) \) is said to be Kenmotsu manifold if

\[
(\nabla_X \phi ) Y = g(\phi X, Y) \xi - \eta (X) \phi Y. \tag{2.4}
\]

From equation (2.4), we have

\[
\nabla_X \xi = X - \eta (X), \tag{2.5}
\]

where \( \nabla \) denotes the Riemannian connection of \( g \).

Let \((g, V, \lambda)\) be a Ricci soliton an \( n \)- dimensional Kenmotsu manifold \( M \).

From equation (2.5), we have

\[
\left( \mathcal{L}_\xi g \right)(X, Y) = 2[g(X, Y) - \eta (X) \eta (Y)], \tag{2.6}
\]

From equations (2.1) and (2.6), we have

\[
S(X, Y) = -(\lambda + 1)g(X, Y) + \eta (X) \eta (Y), \tag{2.7}
\]

\[
S(\phi X, \phi Y) = S(X, Y) + \lambda \eta (X) \eta (Y) \tag{2.8}
\]

From equation (2.8), we have

\[
QX = -(\lambda + 1)X + \eta (X) \xi, \tag{2.9}
\]

\[
S(X, \xi) = -\lambda \eta (X), \tag{2.10}
\]

\[
r = -\lambda n - (n - 1), \tag{2.11}
\]

where \( S \) is the Ricci tensor, \( Q \) is the Ricci operator and \( r \) is the scalar curvature on \( M \).

**Theorem 3.1:** A Concircular curvature tensor \( \tilde{C} \) in Kenmostu manifold satisfying \( R(\xi, X). \tilde{C} = 0 \) is expanding.

**Proof:** Let us suppose \((R(\xi, X), \tilde{C}) (U, V)W = 0\)

\[
\Rightarrow R(\xi, X) \tilde{C} (U, V) W - \tilde{C}(R(\xi, X) U, V) W - \tilde{C}(U, R(\xi, X) V) W - \tilde{C}(U, V) R(\xi, X) W = 0. \tag{3.1}
\]

Using equation (2.2-8) in above equation, we get

\[
\eta(\tilde{C}(U, V) W) X - g(X, \tilde{C}(U, V) W) \xi - \tilde{C}(\eta(U) X - g(X, U) \xi, V) W - \tilde{C}(U, \eta(V) X - g(X, V) \xi) W - \tilde{C}(U, V) (\eta(W) X - (X, W) \xi) = 0, \tag{3.2}
\]

\[
\Rightarrow \eta(\tilde{C}(U, V) W) X - g(X, \tilde{C}(U, V) W) \xi - \tilde{C}(\eta(U) X - g(X, U) \xi, V) W - \tilde{C}(U, \eta(V) X - g(X, V) \xi) W - \tilde{C}(U, V) (\eta(W) X - (X, W) \xi) = 0. \tag{3.3}
\]

Now taking an inner product of above equation with \( \xi \), we get

\[
\Rightarrow \eta(\tilde{C}(U, V) W) \eta(X) - g(X, \tilde{C}(U, V) W) \eta(X) - g(X, \til{C}(U, V) W) \eta(U) \eta(\til{C}(\xi, V) W) + g(X, U) \eta(\til{C}(U, V) X) + g(X, W) \eta(\til{C}(U, V) \xi) = 0. \tag{3.3}
\]

Using equation (2.1.6) and (1.7) in above equation, we get

\[
[1 + \frac{r}{n(n-1)}] [g(U, W) \eta(V) - g(X, W) \eta(U)] \eta(X) - g(X, \til{C}(U, V) W) \eta(U) \left[ 1 + \frac{r}{n(n-1)} \right] [g(X, W) \eta(V) - g(U, W) \eta(U)] \eta(V) \left[ 1 + \frac{r}{n(n-1)} \right] \tag{3.3}
\]
\[ [g(U, W) \eta(X) - g(X, W)\eta(U)] + g(X, V)\left[1 + \frac{r}{n(n-1)}\right] [g(U, W) \eta(\xi) - g(\xi, W)\eta(U)] - \eta(W)\left[1 + \frac{r}{n(n-1)}\right] [g(U, W) \eta(V) - g(V, W)\eta(U)] + g(X, W)\left[1 + \frac{r}{n(n-1)}\right] [g(U, W) \eta(\xi) - g(\xi, W)\eta(U)] = 0, \]

\[ \Rightarrow [1 + \frac{r}{n(n-1)}] [g(U, W) \eta(X)V - g(V, W)\eta(U)\eta(X) - g(X, W) \eta(V)\eta(U)] + g(V, W)\eta(X)\eta(U) + g(X, W) \eta(V)\eta(U) = 0, \]

\[ \Rightarrow X = U = e^i, \] in above equation and summing over \( i \), \( 1 \leq i \leq n \), we get

\[ -g(X, \tilde{C}(U, V)W) + [1 + \frac{r}{n(n-1)}] [g(X, V) g(U, W) - g(X, U) g(V, W)] = 0. \]  \( \text{(3.4)} \)

In view of equation (1.1), above equation takes the form

\[ -g(X, R(U, V)W) - \frac{r}{n(n-1)} [g(X, V) g(U, W) - g(X, U) g(V, W)] + [1 + \frac{r}{n(n-1)}] [g(X, V) g(U, W) - g(X, U) g(V, W)] = 0, \]

\[ \Rightarrow -g(X, R(U, V)W) - \frac{r}{n(n-1)} [g(X, V) g(U, W) - g(X, U) g(V, W)] + [1 + \frac{r}{n(n-1)}] [g(X, V) g(U, W) - g(X, U) g(V, W)] = 0 \]

\[ \Rightarrow g(X, R(U, V)W) + [g(X, V) g(U, W) - g(X, U) g(V, W)] = 0. \]  \( \text{(3.5)} \)

Taking \( X = U = e^i \), in above equation and summing over \( i \), \( 1 \leq i \leq n \), we get

\[ -R(e^i, V, W, e^j) + [g(e^i, V) g(e^j, W) - g(e^j, e^i) g(V, W)] = 0 \]

\[ \Rightarrow -S(V, W) + [g(V, W) - ng(V, W)] = 0. \]  \( \text{(3.6)} \)

Taking \( V = W = \xi \), in above equation, we get

\[ -S(\xi, \xi) + [g(\xi, \xi) - g(\xi, \xi)] = 0 \]

\[ \Rightarrow \lambda + (1 - n) = 0 \]

\[ \Rightarrow \lambda = n - 1. \]

This shows that \( \lambda \) is positive, it means that the concircular curvative tensor \( \tilde{C} \) in Kenmostu manifold satisfying \( R(\xi, X)\tilde{C} = 0 \) is expanding. This completes the proof.

References