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Statistical properties and applications of a Weibull-Kumaraswamy distribution

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Abstract
The Kumaraswamy probability distribution was originally proposed by Poondi Kumaraswamy (1980) for double bounded random processes for hydrological applications. Here, we proposed a new extension of the Kumaraswamy distribution by adding one shape and one scale parameter to the Kumaraswamy distribution using the Weibull link function proposed by Tahir *et al*. (2015). This study has derived some expressions for its basic statistical properties such as moments, moment generating function, the characteristics function, reliability analysis, quantile function and the distribution of order statistics. Some plots of the distribution and the reliability function were generated and interpreted appropriately. The model parameters have been estimated using the method of maximum likelihood estimation. The performance of the Weibull-Kumaraswamy distribution has also been tested by some applications to two real data sets.

Keywords: Weibull-Kumaraswamy distribution, statistical properties, order statistics, reliability analysis, estimation, parameters, application

1. Introduction

The Kumaraswamy probability distribution was originally proposed by Poondi Kumaraswamy (1980) [6] for double bounded random processes for hydrological applications. The Kumaraswamy double bounded distribution denoted by (a, b) distribution is a family of continuous probability distributions defined on the interval [0, 1] with cumulative distribution function (*cdf*) given by

$$G(x) = 1 - \left(1 - x^{a}\right)^{b} \tag{1}$$

And the corresponding probability density function (*pdf*) given by

$$g(x) = ab\, x^{a-1}\left(1 - x^{a}\right)^{b-1} \tag{2}$$

For $0 \le x \le 1$, where $a > 0$ and $b > 0$ are the shape parameters.

We have so many generalized families of distributions proposed by different researchers that are used in extending other distributions to produce compound distributions with better performance. We also have some generalizations of the Kumaraswamy distribution recently proposed in the literature such as the transmuted Kumaraswamy distribution by Khan *et al*., (2016) [5], the exponentiated Kumaraswamy distribution by Javanshiri *et al*. (2015) [3] and the Kumaraswamy- Kumaraswamy distribution by El-Sherpieny and Ahmad (2014) [2].

In the next section, we have obtained the *cdf* and *pdf* of the Weibull-Kumaraswamy distribution *(WKD)* using the steps proposed by Tahir *et al*, (2015) [7]. According to them, the formula for deriving the *cdf* and *pdf* of any Weibull-based distribution from the above Weibull-G family is defined for any continuous distribution as follows:

$$F(x) = \int_{0}^{-\log[G(x)]} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}\, dt = e^{-\alpha\{-\log[G(x)]\}^{\beta}} \tag{3}$$

and

$$f(x) = \alpha\beta \frac{g(x)}{G(x)} \{-\log[G(x)]\}^{\beta-1} e^{-\alpha\{-\log[G(x)]\}^{\beta}} \tag{4}$$

Respectively, where g(*x*) and G(*x*) are the *pdf* and *cdf* of any continuous distribution to be generalized respectively and α>0 and β>0 are the two additional new parameters responsible for the scale and shape of the distribution respectively.

The aim of this article is to introduce a new continuous distribution called Weibull-Kumaraswamy distribution (*WKD)* from the proposed family by Tahir *et al*. (2015) [7]. The remaining parts of this paper are presented in sections as follows: We defined the new distribution and give its plots in section 2. Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. In section 5, we carryout application of the proposed model with others to two real life data sets. Lastly, in section 6, we give some concluding remarks.

2. The Weibull-Kumaraswamy distribution (*WKD)*

Using equation (1) and (2) in (3) and (4) and simplifying, we obtain the *cdf* and *pdf* of the Weibull-Kumaraswamy distribution as follows:

$$F(x) = 1 - e^{-\alpha\left\{-\log\left[1-(1-x^a)^b\right]\right\}^{\beta}}; 0 < x < 1 \tag{5}$$

and

$$f(x) = ab\alpha\beta \frac{x^{a-1}(1-x^a)^{b-1}}{\left[1-(1-x^a)^b\right]} \left\{-\log\left[1-(1-x^a)^b\right]\right\}^{\beta-1} e^{-\alpha\left\{-\log\left[1-(1-x^a)^b\right]\right\}^{\beta}} \tag{6}$$

respectively.

For $0 < x < 1; a, b, \alpha, \beta > 0$, where *a*> 0 and *b*> 0 are the shape parameters while $\alpha > 0 and \beta > 0$ are scale and shape parameters respectively.

Given some values for the parameters *α, β, a* and *b*, we provide some possible shapes for the *pdf* and the *cdf* of the *WKD* as shown in figure 1 and 2 below:

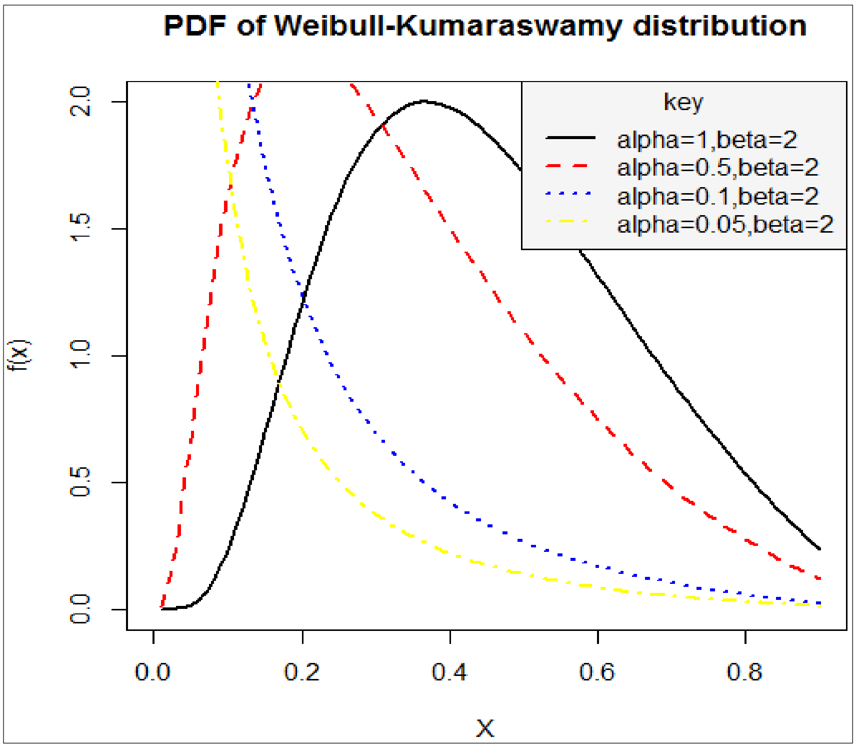

Fig. **1**: *PDF* of the *WKD* for different values of the parameters $\alpha - and - \beta$ where $a = b = 1$ as displayed in the key on the graph.

Figure 1 indicates that the *WKD* distribution has various shapes such as left-skewed or right-skewed shapes depending on the parameter values. This means that distribution can be very useful for datasets with different shapes.

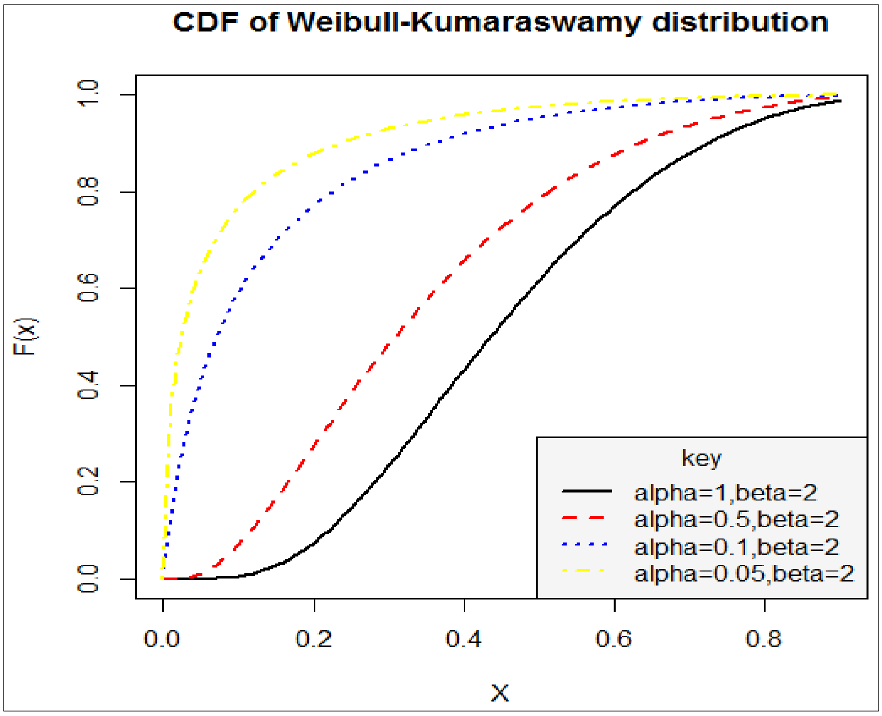

Fig 2: *CDF* of the *WKD* for different values of the parameters $\alpha - and - \beta$ where $a = b = 1$ as displayed in the key on the graph.

From the above *cdf* plot, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3. Properties

In this section, we defined and discuss some properties of the *WKD* distribution.

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left[X^n\right] = \int_0^{\infty} x^n f(x)dx \tag{7}$$

Considering *f(x)* to be the *pdf* of the Weibull-Kumaraswamy distribution as given in equation (7).

$$\mu_n^{'} = E\left[X^n\right] = \int_0^{1} x^n f(x)dx$$

Recall,

$$f(x) = ab\alpha\beta \frac{x^{a-1}\left(1-x^a\right)^{b-1}}{\left[1-\left(1-x^a\right)^b\right]} \left\{-\log\left[1-\left(1-x^a\right)^b\right]\right\}^{\beta-1} e^{-\alpha\left\{-\log\left[1-\left(1-x^a\right)^b\right]\right\}^{\beta}} \tag{8}$$

Let

$$A = e^{-\alpha\left\{-\log\left[1-\left(1-x^a\right)^b\right]\right\}^{\beta}}$$

Then, using a power series expansion for A, we can write A as:

$$A = e^{-\alpha\left\{-\log\left[1-\left(1-x^a\right)^b\right]\right\}^{\beta}} = \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i}{i!} \left(-\log\left[1-\left(1-x^a\right)^b\right]\right)^{\beta i}$$

Substituting for the expansion above in equation (8), we have;

$$f(x)=ab\alpha\beta\frac{x^{a-1}\left(1-x^{a}\right)^{b-1}}{\left[1-\left(1-x^{a}\right)^{b}\right]}\left\{-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right\}^{\beta-1}\sum_{i=0}^{\infty}\frac{(-1)^{i}\alpha^{i}}{i!}\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)^{\beta i}$$

$$=ab\alpha\beta\frac{x^{a-1}\left(1-x^{a}\right)^{b-1}}{\left[1-\left(1-x^{a}\right)^{b}\right]}\sum_{i=0}^{\infty}\frac{(-1)^{i}\alpha^{i}}{i!}\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)^{\beta(i+1)-1} \tag{9}$$

Also, let $B=\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)^{\beta(i+1)-1}$

then, the following formula holds for B for $i\geq 1$ (http://function.wolfram.com/Elementaryfunctions/log/06/01/04/03/), and then we can write the B as follows:

$$\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)^{\beta(i+1)-1}=\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l}\left(\beta(i+1)\right)}{\left(\beta(i+1)-1-j\right)}\binom{k-\left(\beta(i+1)-1\right)}{k}\binom{k}{j}\binom{\left(\beta(i+1)-1\right)+k}{l}P_{j,k}\left[1-\left(1-x^{a}\right)^{b}\right]^{l} \tag{10}$$

Where for (for j≥0) $P_{j,0}=1$ and (for k=1,2,…..)

$$P_{j,k}=k^{-1}\sum_{m=1}^{k}(-1)^{m}\frac{\left[m(j+1)-k\right]}{(m+1)}P_{j,k-m} \tag{11}$$

Combining equation (9) and inserting the above power series in equation (10) and simplifying, we have:

$$f(x)=\alpha\beta\sum_{i=0}^{\infty}\frac{(-1)^{i}\alpha^{i}}{i!}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l}\left(\beta(i+1)\right)}{\left(\beta(i+1)-1-j\right)}\binom{k-\left(\beta(i+1)-1\right)}{k}\binom{k}{j}\binom{\left(\beta(i+1)-1\right)+k}{l}P_{j,k}ab\,x^{a-1}\left(1-x^{a}\right)^{b-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{l-1}$$

$$f(x)=\beta\sum_{i=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l}\alpha^{i+1}\left(\beta(i+1)\right)}{i!\left(\beta(i+1)-1-j\right)}\binom{k-\left(\beta(i+1)-1\right)}{k}\binom{k}{j}\binom{\left(\beta(i+1)-1\right)+k}{l}P_{j,k}ab\,x^{a-1}\left(1-x^{a}\right)^{b-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{l-1} \tag{12}$$

Now, if *l* is a positive non-integer, we can expand the last term in (12) as:

$$\left[1-\left(1-x^{a}\right)^{b}\right]^{l-1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{l-1}{m}\left[\left(1-x^{a}\right)^{b}\right]^{m}$$

Therefore, *f(x)* becomes:

$$f(x)=\beta\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}\alpha^{i+1}\left(\beta(i+1)\right)}{i!\left(\beta(i+1)-1-j\right)}\binom{k-\left(\beta(i+1)-1\right)}{k}\binom{l-1}{m}\binom{k}{j}\binom{\left(\beta(i+1)-1\right)+k}{l}P_{j,k}ab\,x^{a-1}\left(1-x^{a}\right)^{b(m+1)-1}$$

$$=W_{i,j,k,l,m}ab\,x^{a-1}\left(1-x^{a}\right)^{b(m+1)-1} \tag{13}$$

Where $W_{i,j,k,l,m}=\beta\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}\alpha^{i+1}\left(\beta(i+1)\right)}{i!\left(\beta(i+1)-1-j\right)}\binom{k-\left(\beta(i+1)-1\right)}{k}\binom{l-1}{m}\binom{k}{j}\binom{\left(\beta(i+1)-1\right)+k}{l}P_{j,k}$

Hence,

$$\mu_n' = E\left[X^n\right] = \int_0^1 x^n f(x)dx = W_{i,j,k,l,m} ab \int_0^1 x^{n+a-1}\left(1-x^a\right)^{b(m+1)-1} dx \tag{14}$$

Recall the for the Kumaraswamy distribution;

$$E\left[X^r\right] = \int_0^1 x^r f(x)dx = ab\int_0^1 x^{r+a-1}\left(1-x^a\right)^{b-1} dx = bB\left(\tfrac{r}{a}+1, b\right)$$

Hence, this implies that

$$\mu_n' = E\left[X^n\right] = \int_0^1 x^n f(x)dx = W_{i,j,k,l,m} ab \int_0^1 x^{n+a-1}\left(1-x^a\right)^{b(m+1)-1} dx$$

$$= \int_0^1 x^n f(x)dx = W_{i,j,k,l,m} ab \int_0^1 x^{n+a-1}\left(1-x^a\right)^{b(m+1)-1} dx = W_{i,j,k,l,m} bB\left(\tfrac{n}{a}+1, b\left(m+1\right)\right)$$

$$= W_{i,j,k,l,m} bB\left(\tfrac{n}{a}+1, b\left(m+1\right)\right) \tag{15}$$

The Mean

The mean of the *WKD* can be obtained from the n^{th} moment of the distribution when *n=1* as follows:

$$\mu_1' = E\left[X\right] = W_{i,j,k,l,m} bB\left(\tfrac{1}{a}+1, b\left(m+1\right)\right) \tag{16}$$

Also the second moment of the *WKD* is obtained from the n^{th} moment of the distribution when *n=2* as

$$\mu_2' = E\left[X^2\right] = W_{i,j,k,l,m} bB\left(\tfrac{2}{a}+1, b\left(m+1\right)\right) \tag{17}$$

The Variance

The n^{th} central moment or moment about the mean of *X*, say μ_n, can be obtained as

$$\mu_n = E\left[X - \mu_1'\right]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mu_1'^{i} \mu_{n-i}' \tag{18}$$

The variance of *X* for *WKD* is obtained from the central moment when *n=2,* that is,

$$Var(X) = E[X^2] - \left\{E[X]\right\}^2 \tag{19}$$

$$Var(X) = W_{i,j,k,l,m} bB\left(\tfrac{2}{a}+1, b\left(m+1\right)\right) - \left\{W_{i,j,k,l,m} bB\left(\tfrac{1}{a}+1, b\left(m+1\right)\right)\right\}^2 \tag{20}$$

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

3.2 Moment Generating Function

The *mgf* of a random variable *X* can be obtained by

$$M_x(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx} f(x)dx \tag{21}$$

Using power series expansion in equation (21) and simplifying the integral in (21), therefore we have;

$$M_x(t)=E\left[e^{tx}\right]=\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_n^{'}=\sum_{n=0}^{\infty}\frac{t^n}{n!}W_{i,j,k,l,m}bB\left(\tfrac{n}{a}+1,b(m+1)\right) \tag{22}$$

where n and t are constants, t is a real number and μ_n' denotes the n^{th} ordinary moment of X.

3.3 Characteristics Function

The characteristics function of a random variable X is given by;

$$\varphi_x(t)=E\left[e^{itx}\right]=E\left[\cos(tx)+i\sin(tx)\right]=E\left[\cos(tx)\right]+E\left[i\sin(tx)\right] \tag{23}$$

Simple algebra and power series expansion proves that

$$\phi_x(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu_{2n}^{'}+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu_{2n+1}^{'} \tag{24}$$

Where μ_{2n}' and μ_{2n+1}' are the moments of X for $n=2n$ and $n=2n+1$ respectively and can be obtained from μ_n' in equation (15)

3.4 Quantile function for the *WKD*.

The quantile function, say $X=Q(u)$, of the *WKD* can be obtained as the inverse of Equation (5) as;

$$F(x)=1-e^{-\alpha\left\{-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right\}^{\beta}}$$

$$X_q=Q(u)=\left[1-\sqrt[b]{\left(1-\exp\left\{-\left(\frac{1}{\alpha}\ln\left(\frac{1}{1-u}\right)\right)^{\frac{1}{\beta}}\right\}\right)}\right]^{\frac{1}{a}} \tag{25}$$

By using (25) above, the median of X from the *WKD* is simply obtained by setting u=0.5 while random numbers can be generated from *WKD* by setting $X=Q(u)$, where u is a uniform variate on the unit interval *(0,1)*.

3.5 Skewness and kurtosis

The quantile based measures of skewness and kurtosis will employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \tag{26}$$

And the Moores' (1998) kurtosis is on octiles and is given by;

$$KT=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{1}{4}\right)} \tag{27}$$

where $Q(.)$ is any quartile or octile of interest.

3.6 Reliability analysis of the *WKD*.

Survival function is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(X)=P(X>x)=1-F(x) \tag{28}$$

Applying the *WKD* in (28), the survival function for the *WKD* is obtained as:

$$S(x)=e^{-\alpha\left\{-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right\}^{\beta}} \quad (29)$$

Below is a plot of the survival function at chosen parameter values in figure 3

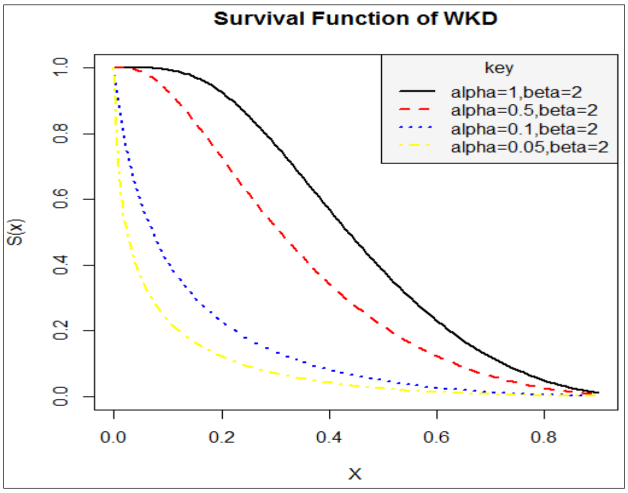

Fig 3: The survival function of the *WKD* for different values of the parameters $\alpha - and - \beta$ where $a=b=1$ as displayed in the key on the graph.

The figure above revealed that the probability of survival for any random variable following a Weibull-Kumaraswamy distribution reduces as the values of the random variable becomes larger, that is, as age grows, probability of life decreases. This implies that the Weibull-Kumaraswamy distribution can be used to model random variables whose survival rate decreases as their age grows where *0<x<1*.

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)} \quad (30)$$

Substituting for *f(x)* and *F(x)* and simplifying gives

$$h(x)=\frac{ab\alpha\beta x^{a-1}\left(1-x^{a}\right)^{b-1}\left\{-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right\}^{\beta-1}}{\left[1-\left(1-x^{a}\right)^{b}\right]} \quad (31)$$

The following is a plot of the hazard function at chosen parameter values in figure 4

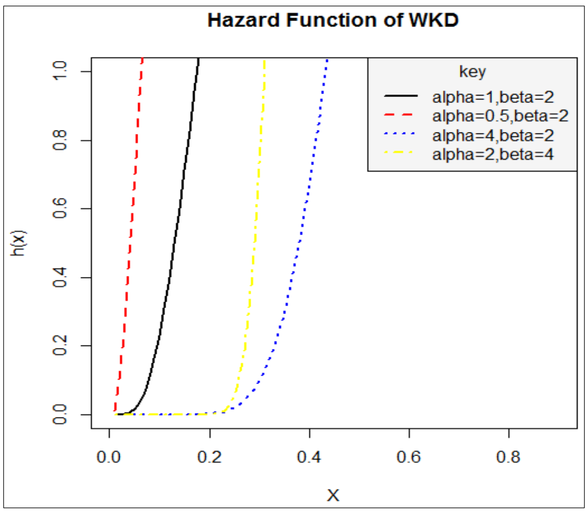

Fig 4: The hazard function of the *WKD* for different values of the parameters $\alpha - and - \beta$ where $a=b=1$ as displayed in the key on the graph.

Interpretation: the figure above revealed that the probability of failure for any random variable following a Weibull-Kumaraswamy distribution increases as the values of the random variable increases, that is, as time goes on, probability of death increases. This implies that the Weibull-Kumaraswamy distribution can be used to model random variables whose failure rate increases as their age grows.

3.7 Order Statistics

In this section, we derive closed form expressions for the *pdf* of the a^{th} order statistics of the *WKD*. If $X_1, X_2, ..., X_n$ is a random sample from the *WKD* and also let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ be the corresponding order statistic obtained from this sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \tag{32}$$

where *f(x)* and *F(x)* are the *pdf* and *cdf* of the proposed distribution respectively.

Using (5) and (6), the *pdf* of the i^{th} order statistics$X_{i:n}$, can be expressed from as;

(33)

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[W_{i,j,k}ab\chi^{a-1}\left(1-x^a\right)^{b-1}\left[1-\left(1-x^a\right)^b\right]^{l-1}\right]*\left[1-\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left(-\log\left[1-\left(1-x^a\right)^b\right]\right)^{\beta i}\right]^{i+k-1} \tag{34}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *WKD* are given by;

$$f_{1:n}(x) = n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[W_{i,j,k}ab\chi^{a-1}\left(1-x^a\right)^{b-1}\left[1-\left(1-x^a\right)^b\right]^{l-1}\right]*\left[1-\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left(-\log\left[1-\left(1-x^a\right)^b\right]\right)^{\beta i}\right]^{k} \tag{35}$$

And

$$f_{n:n}(x) = n\left[W_{i,j,k}ab\chi^{a-1}\left(1-x^a\right)^{b-1}\left[1-\left(1-x^a\right)^b\right]^{l-1}\right]\left[1-\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left(-\log\left[1-\left(1-x^a\right)^b\right]\right)^{\beta i}\right]^{n-1} \tag{36}$$

respectively.

4. Estimation of Parameter

Let $X_{1, - - -,}X_n$ be a sample of size 'n' independently and identically distributed random variables from the *WKD* with unknown parameters *α, β, a,* and *b* defined previously. The likelihood function is given by;

$$L\left(X_1, X_2, ..., X_n / a, b, \alpha, \beta\right) = \frac{\left(ab\alpha\beta^{\alpha}\right)^n \sum_{i=1}^{n} x_i^{a-1}\sum_{i=1}^{n}\left(1-x_i^a\right)^{b-1}}{\sum_{i=1}^{n}\left[1-\left(1-x_i^a\right)^b\right]}\sum_{i=1}^{n}\left(-\log\left[1-\left(1-x_i^a\right)^b\right]\right)^{\beta-1} e^{-\alpha\sum_{i=1}^{n}\left\{-\log\left[1-\left(1-x_i^a\right)^b\right]\right\}^{\beta}} \tag{37}$$

Let the log-likelihood function, $l = \log L\left(X_1, X_2, ..., X_n / \alpha, \beta, a, b\right)$, therefore

$$l = n\log\alpha + n\alpha\log\beta + n\log a + n\log b + (a-1)\sum_{i=1}^{n}\log x_i + (b-1)\sum_{i=1}^{n}\log\left(1-x_i^a\right) - \sum_{i=1}^{n}\log\left[1-\left(1-x_i^a\right)^b\right]$$

$$+(\beta-1)\sum_{i=1}^{n}\log\left(-\log\left[1-\left(1-x_i^a\right)^b\right]\right) - \alpha\sum_{i=1}^{n}\left\{-\log\left[1-\left(1-x_i^a\right)^b\right]\right\}^{\beta} \tag{38}$$

Differentiating l partially with respect to *α, β, a* and *b* respectively gives;

$$\frac{\partial l}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log x_i+(b-1)\sum_{i=1}^{n}\left\{\frac{x_i^{a}\ln x_i}{\left(1-x_i^{a}\right)}\right\}-b\sum_{i=1}^{n}\left\{\frac{\left(1-x_i^{a}\right)^{b-1}x_i^{a}\ln x_i}{\left[1-\left(1-x_i^{a}\right)^{b}\right]}\right\}+b(\beta-1)\sum_{i=0}^{n}\left\{\frac{\left(1-x_i^{a}\right)^{b-1}x_i^{a}\ln x_i}{\left(-\log\left[1-\left(1-x_i^{a}\right)^{b}\right]\right)\left[1-\left(1-x_i^{a}\right)^{b}\right]}\right\}$$
$$+\alpha b\sum_{i=0}^{n}\left\{\frac{\left(-\log\left[1-\left(1-x_i^{a}\right)^{b}\right]\right)^{\beta-1}\left(1-x_i^{a}\right)^{b-1}x_i^{a}\ln x_i}{\left[1-\left(1-x_i^{a}\right)^{b}\right]}\right\} \tag{39}$$

$$\frac{\partial l}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log x_i+b\sum_{i=1}^{n}\log\left(1-x_i^{a}\right)-\sum_{i=1}^{n}\left\{\frac{\left(1-x_i^{a}\right)^{b}\ln\left(1-x_i^{a}\right)}{\left[1-\left(1-x_i^{a}\right)^{b}\right]}\right\}-(\beta-1)\sum_{i=0}^{n}\left\{\frac{\left(1-x_i^{a}\right)^{b}\ln\left(1-x_i^{a}\right)}{\left(-\log\left[1-\left(1-x_i^{a}\right)^{b}\right]\right)\left[1-\left(1-x_i^{a}\right)^{b}\right]}\right\}$$
$$-\alpha\sum_{i=0}^{n}\left\{\frac{\left(-\log\left[1-\left(1-x_i^{a}\right)^{b}\right]\right)^{\beta-1}\left(1-x_i^{a}\right)^{b}\ln\left(1-x_i^{a}\right)}{\left[1-\left(1-x_i^{a}\right)^{b}\right]}\right\} \tag{40}$$

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)^{\beta} \tag{41}$$

$$\frac{\partial l}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)-\alpha\sum_{i=1}^{n}\log\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right)^{\beta}\log\left(-\log\left[1-\left(1-x^{a}\right)^{b}\right]\right) \tag{42}$$

The solution of the non-linear system of equations; $\frac{\partial l}{\partial a}=0$, $\frac{\partial l}{\partial b}=0$, $\frac{dl}{d\alpha}=0,\ and\ \frac{\partial l}{\partial \beta}=0,$ will give us the maximum likelihood estimates of parameters a,b,α and β. However, the solution cannot be gotten analytically except numerically with the aid of suitable statistical software like R, SAS, e.t.c when data sets are available.

5. Applications

This section presents applications of the two datasets to some selected generalizations of the Kumaraswamy distribution. We have compared the performance of the Weibull-Kumaraswamy distribution (*WKD*) to those of three models such as the transmuted Kumaraswamy distribution (*TKD*), the Kumaraswamy-Kumaraswamy distribution (*KKD*) and the Kumaraswamy distribution (*KD*). The two datasets are as follows:

Data set I: This data is flood data with 20 observations obtained from Dumonceaux and Antle (1973) and it has been used by Khan *et al.* (2016) [5].

Data set II: The second data set is on shape measurements of 48 rock samples from a petroleum reservoir. This data was extracted from BP research, image analysis by Ronit Katz, u Oxford and has been used for analysis by Javanshiri *et al*. (2015) [3].
The summary of the two data sets is also provided in Table 1 as follows;

Table 1: Summary Statistics for the two data sets

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values for data set I	20	0.265	0.3345	0.4070	0.4578	0.4232	0.7400	0.0157	1.0677	0.5999
Values for data set II	48	0.0903	0.1623	0.1988	0.2627	0.2181	0.4641	0.0069	1.1694	1.1099

We also provide some histograms and densities for the two data sets as shown in Figure 5 and 6 below respectively.

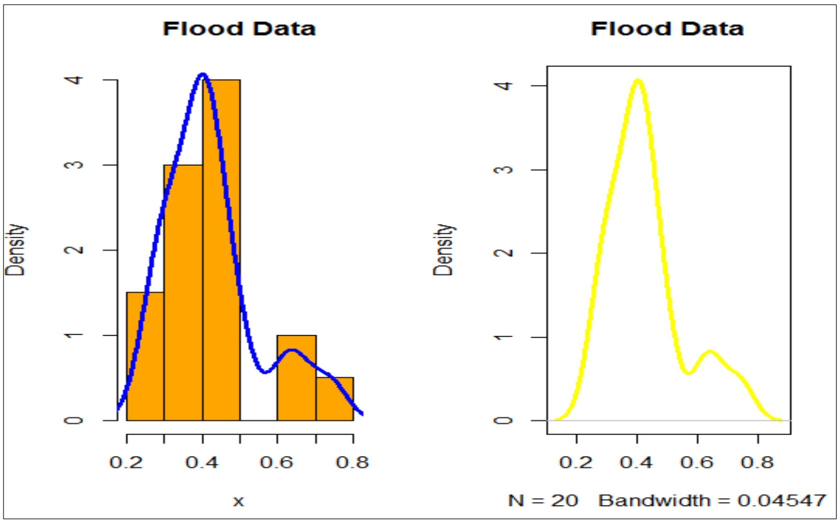

Fig 5: A histogram and density plot for the flood data (Data set I)

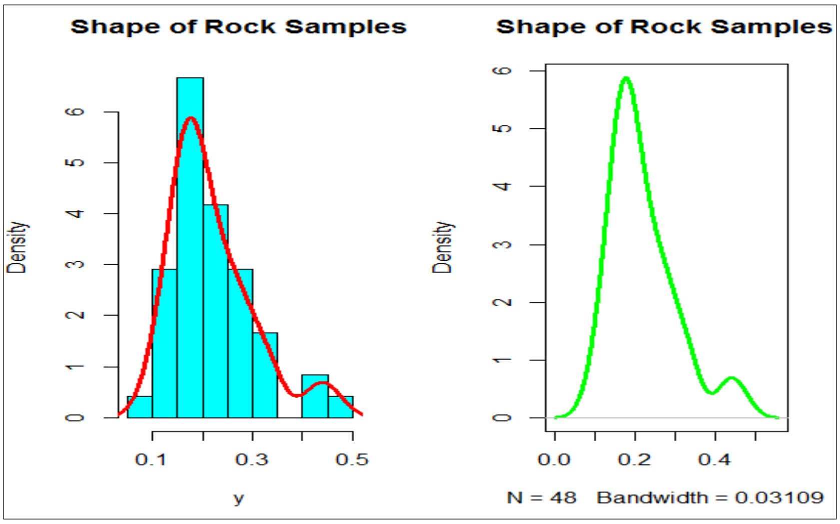

Fig 6: A Histogram and density plot for the rock sample data (Data set II)

From the descriptive statistics, histograms and densities shown above for the two data sets, we observed that both the first and second data sets are positively skewed and therefore suitable for distributions that are skewed to the right.
For us to assess the models listed above, we made use of some criteria: the *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *HQIC* (Hannan Quin information criterion).
Decision benchmark: The model with the lowest values of these statistics would be chosen as the best model to fit the data.
The estimates and performance measures of the above distributions are listed in table 2 and 3 for datasets I and II respectively as follows:

Table 2: Performance of the distribution using the *AIC, CAIC, BIC and HQIC* values of the models based on data set I.

Distributions	**Parameter estimates**	**-ll=(-log-likelihood value)**	***AIC***	***CAIC***	***BIC***	***HQIC***	**Ranks of models performance**
WKD	$\hat{a}$=0.7912 $\hat{b}$=1.2937 $\hat{\alpha}$=5.3826 $\hat{\beta}$=3.2498	-15.9785	-23.9569	-21.2902	-19.9739	-23.3794	1
TKD	$\hat{a}$=2.3594 $\hat{b}$=9.3721 $\hat{\lambda}$=-0.9873	-13.7230	-21.4461	-19.9460	-18.4589	-20.8629	4
KKD	$\hat{a}$=1.4228 $\hat{b}$=4.2086 $\hat{\alpha}$=4.6539 $\hat{\beta}$=1.8144	-14.0841	-20.1682	-17.5015	-16.1852	-19.3907	5
EKD	$\hat{a}$=1.0795 $\hat{b}$=5.3019 $\hat{\gamma}$=9.1059	-14.9333	-23.8667	-22.3667	-20.8795	-23.2836	2
KD	$\hat{a}$=3.1247 $\hat{b}$=9.5689	-12.7643	-21.5287	-20.8227	-19.5372	-21.1399	3

From Table 2, the values of the parameter MLEs and the corresponding values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* for each model shows that the *WKD* has better performance compared to the *KKD*, *EKD*, *TKD* and *KD*. It also agrees with the fact that generalizing any continuous distribution provides a compound distribution with a better fit than the classical distribution in the view that the *WKD* distribution outperformed the classical *KD* itself.

Table 3: Performance of the distribution using the *AIC, CAIC, BIC and HQIC* values of the models based on data set II.

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models performance
WKD	$\hat{a}$=1.0850 $\hat{b}$=5.0029 $\hat{\alpha}$=3.9708 $\hat{\beta}$=2.2984	-57.8913	- 107.7825	- 106.8523	- 100.2977	-104.954	1
TKD	$\hat{a}$=1.4353 $\hat{b}$=9.8929 $\hat{\lambda}$=-0.9724	-50.3032	-94.6064	-94.0609	-88.9928	-92.4851	4
KKD	$\hat{a}$=0.8379 $\hat{b}$=4.3626 $\hat{\alpha}$=5.7753 $\hat{\beta}$=3.3596	-55.2763	- 102.5526	- 101.6224	-95.0678	-99.7241	3
EKD	$\hat{a}$=0.9354 $\hat{b}$=9.8799 $\hat{\gamma}$=6.7262	-12.7643	- 102.6815	-102.136	- 97.06786	- 100.5601	2
KD	$\hat{a}$=1.6673 $\hat{b}$=9.4586	-44.6096	-85.2193	-84.9526	-81.4769	-83.8049	5

Table 2 also shows the parameter estimates to each one of the five fitted distributions for the second data set (data set II), the table also provide the values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* of the fitted models. The values in Table 3 also indicate that the Weibull-Kumaraswamy distribution has better performance with the lowest values of *AIC*, *CAIC*, *BIC* and *HQIC*. The *WKD* performed better than the *KKD, TKD, EKD* and *KD*.

6. Conclusion

This study has proposed a new distribution. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done appropriately. Some plots of the distribution revealed that it is a positively skewed distribution. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the Weibull-Kumaraswamy distribution could be used to model time or age-dependent events, where survival rate decreases with time or age. The performance of the new distribution is illustrated by some applications to two real data sets. The results showed that the new distribution, *WKD* performs better than the Kumaraswamy-Kumaraswamy, Transmuted Kumaraswamy, Exponentiated Kumaraswamy and the Kumaraswamy distributions as shown in the analysis for data set I and II in section 5.

7. References

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