International Journal of Statistics and Applied Mathematics 2019; 4(1): 01-04

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2019; 4(1): 01-04

www.mathsjournal.com
Received: 01-11-2018
Accepted: 05-12-2018

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K- Trianalytic functions

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Abstract
In this paper, the concept of K- trianalytic functions on complex plane is presented and the expression of K- trianalytic functions is given. Then, we proved the integral theorem. Also, the power series expansion theorem was proved in this study.

Keywords: K- trianalytic functions; integral theorem; power series

1. Introduction

Analytic functions have been applied in many fields as a powerful tool, e.g., theoretical physics, astromechanics, fluid mechanics and elastic mechanics. Over the past two hundred years, systematic theories in analytic functions have been established through the efforts of many researchers [1-5]. Recently, some researchers have proposed the concept of K-analytic function and studied its analytical properties (See [6-9]). Zhang Jianyuan [6] concluded a necessary and sufficient conditions of K-analytic function. In [7], the authors deducts the relation between K-analytic function and K-integral, the relation between K-analytic function and K-harmonic function. Zhang jianyuan and his collaborators [8] used the series theory to propose the power series expansion of the K-analytic function and its zero isolation and uniqueness. Then, in [9], the Fourier series of K-analytic function is given and proved. In [10], the authors explained the concept of K- bianalytic function. The function $f(z)$ that satisfies the condition $\frac{\partial^2 f}{\partial \bar{z}^2(k)} = 0$ is called the K- bianalytic function. The K- bianalytic function expression and analysis properties are derived, and the K- bianalytic function series forms are given.

In this research, we explained the concept of K- trianalytic function in the complex plane, and we discussed the properties of K-trianalytic functions. And we derived the K-trianalytic functions expression. Then, we proved the integral theorem of K- trianalytic function. In the last part, we discussed the power series expansion of K-trianalytic functions.

2. Preparatory knowledge

Definition 1[6] (definition of K- complex number) we say that the form $z(k) = x + iky(k \in R, k \neq 0)$ is k-complex number of $z = x + iy$

Definition 2[6] (definition of K-analytic function) If function $f(z)$ is k-differentiable in region D ([6]), $f(z)$ is K-analytic in regionD. All of K-analytic functions in area D is denoted as $F\left(D_1(k)\right)$.

Lemma 1[6] (Necessary and sufficient conditions of K-analytic) the function $f(z) = u(x, y) + iv(x, y)$ is K-analytic in regionD, $\Leftrightarrow$ u, v is differentiable in regionD, and the condition of C-R-K holds,

Where condition C-R-K is: $\begin{cases} u_x = v_y/k \\ v_x = -u_y/k \end{cases}$.

Let $\frac{\partial}{\partial \bar{z}(k)} = \frac{\partial}{\partial x} + \frac{i}{k}\frac{\partial}{\partial y}$, then the complex form of function C-R-K is $\frac{\partial f}{\partial \bar{z}(k)} = 0$.

Lemma 2 [7] The function $\mathrm{f(z)}$ is K-analytic in the single connected region D of z plane, and C is any closed curve in D, then we have the identity

$$\oint_C f(z)dz(k) = 0.$$

Lemma 3 [8] (power series expansion of K-analytic) suppose function $\mathrm{f(z)}$ is K-analytic in regionD, and $B(k): |(z-a)(k)| < R \subset D$, we have

$$f(z) = \sum c_n (z-a)^n(k), z \in B(k),$$

Where

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}(k)} d\zeta(k) = \frac{f^n(a)}{n!},$$

$\left(\Gamma_\rho: |(z-a)(k)| = \rho, 0 < \rho < R, n = 0,1,2,\cdots\right)$, and the expansion is unique.

3. K-trianalytic functions

Definition 3 (definition of K-trianalytic function) the function $f(z)$ has the three derivative $\frac{\partial^3 f}{\partial \bar{z}^3(k)}$ of $\bar{z}(k)$ in the regionD. If a given function $f(z)$ satisfies

$$\frac{\partial^3 f}{\partial \bar{z}^3(k)} = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial^2 f}{\partial \bar{z}^2(k)}\right) = 0,$$

Then $f(z)$ is called K- trianalytic function inD. We use $F(D_3(k))$ to represent the set of all K- trianalytic functions inD.
For the above definition we have the following four propositions.

Proposition 1 If$f(z) \in F\left(D_3(k)\right)$, then$\frac{\partial f}{\partial \bar{z}(k)} \in F(D_2(k))$, and vice versa.

Proposition 2 If$f(z) \in F\left(D_3(k)\right)$, then$\frac{\partial^2 f}{\partial \bar{z}^2(k)} \in F(D_1(k))$, and vice versa.

Proposition 3 Both set of the K-analytic functions and set of the K-analytical functions are a subset of the K-trianalytic functions, that is $F(D_1(k)) \subset F(D_2(k)) \subset F(D_3(k))$.

Proposition 4 If function$f(z) \in F(D_3(k))$, and $\varphi(z)$ is an arbitrary K- analytic function, then $f(z)\varphi(z)$ is K- trianalytic in D.

Proof: Under assumptions we can know $\frac{\partial^3 f}{\partial \bar{z}^3(k)} = 0$, and $\frac{\partial \varphi}{\partial \bar{z}(k)} = 0$, we conclude that

$$\frac{\partial}{\partial \bar{z}(k)}\left(f(z)\varphi(z)\right) = \frac{\partial f}{\partial \bar{z}(k)}\varphi(z) + f(z)\frac{\partial \varphi}{\partial \bar{z}(k)} = \frac{\partial f}{\partial \bar{z}(k)}\varphi(z)$$

Thus there is

$$\frac{\partial^2}{\partial \bar{z}^2(k)}\left(f(z)\varphi(z)\right) = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial f}{\partial \bar{z}(k)}\varphi(z)\right) = \frac{\partial^2 f}{\partial \bar{z}^2(k)}\varphi(z),$$

And

$$\frac{\partial^3}{\partial \bar{z}^3(k)}\left(f(z)\varphi(z)\right) = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial^2 f}{\partial \bar{z}^2(k)}\varphi(z)\right) = \frac{\partial^3 f}{\partial \bar{z}^3(k)}\varphi(z) + \frac{\partial^2 f}{\partial \bar{z}^2(k)}\frac{\partial \varphi}{\partial \bar{z}(k)} = 0.$$

Then we can know that $f(z)\varphi(z)$ is K- trianalytic inD. ■

Theorem 1 (expansion of K- trianalytic functions) If function$f(z) \in F(D_3(k))$, then the following is established

$$f(z) = \sum_{m=0}^{2} \bar{z}^m(k) \cdot \varphi_m(z), (1)$$

Where $\varphi_m(z)(m = 0,1,2)$ is arbitrary K-analytic function in regionD.

Proof: Let$\varphi_2(z) = \frac{\partial^2 f}{\partial \bar{z}^2(k)}$, from Proposition 2 we know that $\varphi_2(z)$ is K-analytic in regionD.

Let$g_2(z)=\frac{1}{2}\bar{z}^2(k)\cdot\varphi_2(z)$, then$\frac{\partial^3 g_2}{\partial\bar{z}^3(k)}=0$, i.e., $g_2(z)$ is a K- trianalytic function inD.

And we can get$\frac{\partial^2 g_2}{\partial\bar{z}^2(k)}=\varphi_2(z)$.

So$\frac{\partial^2(f-g_2)}{\partial\bar{z}^2(k)}=0$.

Let$\varphi_1(z)=\frac{\partial(f-g_2)}{\partial\bar{z}(k)}$, obviously $\varphi_1(z)$ is a K-analytic function inD.

Let$g_1(z)=\bar{z}(k)\cdot\varphi_1(z)$, then $g_1(z)$ is a K- bianalytic function inD, and$\frac{\partial g_1}{\partial\bar{z}(k)}=\varphi_1(z)$.

Then we have$\frac{\partial(f-g_2-g_1)}{\partial\bar{z}(k)}=0$. Thus there is

$$f(z)=g_2+g_1+\varphi_0=\sum_{m=0}^{2}\bar{z}^m(k)\cdot\varphi_m,$$

Where $\varphi_m(z)(m=0,1,2)$ is K-analytic functions in regionD. ■

Theorem 2 (integral theorem of K- trianalytic functions) if f(z) be K- trianalytic in the regionD, and D is a bounded area surrounded by a closed curve C, and f(z),$\frac{\partial f}{\partial\bar{z}(k)},\frac{\partial^2 f}{\partial\bar{z}^2(k)}$ are continuous on$\bar{D}=D+C$, then

$$\int_C\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}\right)dz(k)=0,(2)$$

Or

$$\int_C f(z)dz(k)=\int_C\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}dz(k)-\frac{1}{2}\int_C\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}dz(k),(3)$$

Where a is an arbitrary point in complex plane?

Proof: According to the definition of K- trianalytic function: $\frac{\partial^3 f}{\partial\bar{z}^3(k)}=0$, we have

$$\frac{\partial}{\partial\bar{z}(k)}\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}\right)$$

$$=\frac{\partial f}{\partial\bar{z}(k)}-\left(\frac{\partial f}{\partial\bar{z}(k)}+\overline{z-a}(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}\right)+\left(\overline{z-a}(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^3 f}{\partial\bar{z}^3(k)}\right)$$

$$=0$$

Therefore $f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}$ is K- analytic in D. Under the above assumptions we can know $f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}$ is continuous on $\bar{D}$. From Lemma 2, (2) and (3) are established. ■

Corollary Let the region D be a bounded n+1 connected region surrounded by segmented smooth curve$C=C_0+C_1^-+C_2^-+\cdots+C_n^-$, if function f(z) is K- trianalytic in D, and f(z),$\frac{\partial f}{\partial\bar{z}(k)}$ are continuous on $\bar{D}=D+C$, then

$$\int_C\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}\right)dz(k)=0,$$

Or

$$\int_{c_0}\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}\right)dz(k)$$

$$=\left(\int_{c_1}+\int_{c_2}+\cdots+\int_{c_n}\right)\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}+\frac{1}{2}\overline{z-a}^2(k)\frac{\partial^2 f}{\partial\bar{z}^2(k)}\right)dz(k),$$

Where a is an arbitrary point in complex plane? ■

Theorem 3 (power series of K- trianalytic functions) If $f(z)$ is a K- trianalytic function in region D, and $B(k): |(z-a)(k)| < R \subset D$, for$\forall a \in D$, then $f(z)$ can be expanded into power series in D

$$f(z) = \sum_{n=0}^{+\infty} \sum_{m=0}^{2} c_{nm} \bar{z}^m(k) \cdot (z-a)^n(k)\,, z \in B(k), (3)$$

Where

$$c_{nm} = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\varphi_m}{(\zeta - a)^{n+1}(k)} d\zeta(k), \quad (n = 0,1,2 \cdots), (4)$$

$$\left(\Gamma_\rho: |(z-a)(k)| = \rho, 0 < \rho < R, n = 0,1,2, \cdots\right),$$

And the expansion is unique.

Proof: From theorem 1, we have (1), where $\varphi_m(z)$ is a K-analytic function inD. The following formula obtained by the power series expansion of the K-analytic function (Lemma 3)

$$\varphi_m(z) = \sum_{n=0}^{+\infty} c_{nm} (z-a)^n(k)\,, z \in B(k), m = 0,1,2. (5)$$

Substituting (5) into (1), we have

$$f(z) = \sum_{n=0}^{+\infty} \sum_{m=0}^{2} c_{nm} \bar{z}^m(k) \cdot (z-a)^n(k)\,, z \in B,$$

Where

$$c_{nm} = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\varphi_m}{(\zeta - a)^{n+1}(k)} d\zeta(k), \quad (n = 0,1,2 \cdots).$$

It is obvious that the expansion (3) is unique from the uniqueness of the power series expansion of the K-analytic function. Theorem is proved. ■

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