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A proof of the matrices property $(AB)^* = B^* A^*$

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Abstract

On the basis of the discussion of predecessors on the matrices' property $(AB)^* = B^* A^*$, this article gives a proof which is relatively strict and completely new.

Keywords: Matrices, square matrices of size $n \times n$, property

1. Introduction

Concerning the matrices' properties $(AB)^* = B^* A^*$, a proof is prevalent at present, which is as follows[1]:

Proof. If A, B are two matrices of size $n \times n$

1）When $|A| \neq 0$ and $|B| \neq 0$, $\because (AB)(AB)^* = |AB|E$
$\therefore (AB)^* = |AB|(AB)^{-1} = |A||B|B^{-1}A^{-1} = |B|B^{-1}|A|A^{-1} = A^* B^*$

2）When $|A| = 0$ and $|B| = 0$, let $A(x) = xE + A$, $B(x) = xE + B$

$\therefore$ When x is sufficiently large, $A(x)$, $B(x)$ are both invertible,
$\therefore (A(x)B(x))^* = (B(x))^* (A(x))^*$ $\therefore (AB)^* = B^* A^*$

The entries in the formulas above are all the polynomials about x, so the formula above makes sense for all x. Particularly let $x = 0$, then $(AB)^* = B^* A^*$.

3）When one of $|A|$, $|B|$ is equal to 0 and another is unequal to 0. So we might let $|A| = 0$, $|B| \neq 0$, then use the method in step (2).
Let $A(x) = xE + A$. $\therefore$ when x is sufficiently large, $A(x)$ is invertible.
$\therefore (A(x)B(x))^* = (B(x))^* (A(x))^*$

Similarly let $x = 0$, then $(AB)^* = B^* A^*$

The proof above has some problems actually. In the steps (2) and (3), we let x sufficiently large at first to explain that $A(x)$, $B(x)$ are invertible, but then we let $x = 0$ so that we prove the proposition. In this progress, x is sometimes a large number, and sometimes 0, which is obviously illogical.

2. Preparation

This article is intended to give a rigorous proof.

We firstly give some essential conclusions as follows:

We define the matrix that comes from an identity matrix by adding k times of its i^{th} column to its j^{th} column as the third primary matrix[2], and its notation is $P(i, j(k))$.

That is $P(i, j(k)) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & k & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$

The following results are easily to prove:

$P(i, j(k))^{-1} = P(i, j(-k))$, $P(i, j(k))' = P(j, i(k))$

From $P(i, j(k))^* = |P(i, j(k))| P(i, j(k))^{-1}$, we can obtain that $P(i, j(k))^* = P(i, j(-k))$

Lemma 1: If a square matrix A is of the size $n \times n$, then $(A^*)' = (A')^*$

Proof. If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$, and the algebraic cofactor of a_{ij} is A_{ij}

$$\therefore A^* = \begin{bmatrix} A_{11} & A_{21} & \cdots & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & \cdots & A_{nn} \end{bmatrix} \quad \therefore (A^*)' = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & \cdots & A_{nn} \end{bmatrix}$$

$$\because A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn} \end{bmatrix} \quad \therefore (A')^* = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & \cdots & A_{nn} \end{bmatrix}$$

$\therefore (A^*)' = (A')^*$

Lemma 2: If a matrix A is of the size $n \times n$ $(n \geq 2)$, $P(i, j(k))$ is a third primary matrix of the same size of $n \times n$, then $(AP(i, j(k)))^* = P(i, j(k))^* A^*$, $(P(i, j(k))A)^* = A^* P(i, j(k))^*$

Proof. If $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$ and the confactor and algebraic cofactor of a_{ij} are M_{ij} and A_{ij} respectively.

$$\therefore AP(i,j(k))=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & ka_{1i}+a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & ka_{2i}+a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \cdots & & \vdots & & \vdots \\ \vdots & \vdots & & \cdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & ka_{ni}+a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

If the algebraic cofactor of the entry at the place of intersection of the r^{th} row and the t^{th} column of $AP(i,j(k))$ is $\overline{A_{rt}}$

Then when $t\neq i$, $\overline{A_{rt}}=A_{rt}$

When $t=i$, $\overline{A_{rt}}=(-1)^{r+t}\left[(-1)^{t-1-r}kM_{rj}+M_{rt}\right]=A_{rt}-kA_{rj}$

$$\therefore (AP(i,j(k)))^{*}=\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1i}-kA_{1j} & A_{2i}-kA_{2j} & \cdots & A_{ni}-kA_{nj} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$=P(i,j(-k))A^{*}=P(i,j(k))^{*}A^{*}$$

Additionally $(P(i,j(k))A)^{*}=\left(\left(A'P(i,j(k))'\right)'\right)^{*}=\left((A'P(j,i(k)))'\right)^{*}=\left((A'P(j,i(k)))^{*}\right)'$

$$=\left(P(j,i(k))^{*}(A')^{*}\right)'=\left((A')^{*}\right)'\left(P(j,i(k))^{*}\right)'=A^{*}\left(P(j,i(k))^{*}\right)'=A^{*}P(i,j(k))^{*}$$

$\therefore$ The lemma 2 is proved.

Lemma 3: if $D=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ is a diagonal matrix of size $n\times n$, and A is a matrix of the same size $n\times n$ $(n\geq 2)$, then $(AD)^{*}=D^{*}A^{*}$

Proof. If $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$

$$DA=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}=\begin{bmatrix} d_1a_{11} & d_1a_{12} & \cdots & d_1a_{1n} \\ d_2a_{21} & d_2a_{22} & \cdots & d_2a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_na_{n1} & d_na_{n2} & \cdots & d_na_{nn} \end{bmatrix}$$

$$\therefore (DA)^*=\begin{bmatrix} d_2d_3\cdots d_nA_{11} & d_1d_3\cdots d_nA_{21} & \cdots & d_1d_2\cdots d_{n-1}A_{n1} \\ d_2d_3\cdots d_nA_{12} & d_1d_3\cdots d_nA_{22} & \cdots & d_1d_2\cdots d_{n-1}A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ d_2d_3\cdots d_nA_{1n} & d_1d_3\cdots d_nA_{2n} & \cdots & d_1d_2\cdots d_{n-1}A_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} d_2d_3\cdots d_n & & & \\ & d_1d_3\cdots d_n & & \\ & & \ddots & \\ & & & d_1d_2\cdots d_{n-1} \end{bmatrix}=A^*D^*$$

It can be proved in the same way that $(AD)^*=D^*A^*$

3. The proof of the property

Theorem: If matrix A, B are of the size $n\times n\ (n\geq 2)$, then $(AB)^*=B^*A^*$

Proof. If $A=T_1T_2\cdots T_rD_1T_{r+1}\cdots T_m$, $B=P_1P_2\cdots P_kD_2P_{k+1}\cdots P_s$

Among the matrices above $T_i(i=1,2\cdots m)$ and $P_j(j=1,2\cdots s)$ are all the third primary matrices, D_1, D_2 are both diagonal matrices.

$$\therefore (AB)^*=(T_1T_2\cdots T_rD_1T_{r+1}\cdots T_mP_1P_2\cdots P_kD_2P_{k+1}\cdots P_s)^*$$
$$=(T_2T_3\cdots T_rD_1T_{r+1}\cdots T_mP_1P_2\cdots P_kD_2P_{k+1}\cdots P_s)^*T_1^*$$
$$=(D_1T_{r+1}\cdots T_mP_1P_2\cdots P_kD_2P_{k+1}\cdots P_s)^*T_r^*T_{r-1}^*\cdots T_2^*T_1^*$$
$$=(P_1P_2\cdots P_kD_2P_{k+1}\cdots P_s)^*T_m^*T_{m-1}^*\cdots T_{r+1}^*D_1^*T_r^*\cdots T_2^*T_1^*$$
$$=P_s^*P_{s-1}^*\cdots P_{k+1}^*D_2^*P_k^*\cdots P_2^*P_1^*T_m^*T_{m-1}^*\cdots T_{r+1}^*D_1^*T_r^*\cdots T_2^*T_1^*$$
$$=(P_1P_2\cdots P_kD_2P_{k+1}\cdots P_s)^*(T_1T_2\cdots T_rD_1T_{r+1}\cdots T_m)^*$$
$$=B^*A^*\ \therefore (AB)^*=B^*A^*$$

4. References

1. Yan JP, Guo HX. The Discussion on the Properties of Adjoint Matrices $(AB)^*=B^*A^*$. China's Extracurricular Education, 2014; (21):86.
2. Wang EF, Shi SM. Advanced Algebra. Beijing: Higher Education Press, 188.