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How to construct an auxiliary function to prove proble ms related to differential mean value theorem

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Abstract
This article gives a regular and practical method to prove a kind of problems related to differential mean value theorem which need to construct an auxiliary function.

Keywords: Differential mean value theorem, the construction of an auxiliary function, proof of problems

1. Introduction
In Mathematical Analysis, there are a kind of problems related to differential mean value theorem which need to construct an auxiliary function to prove [1, 2]. However, constructing an auxiliary function is difficult for that the known conditions are sometimes insufficient, and thus they usually waste a long time to be proved. Is there no regular method to construct auxiliary functions for these problems? Actually, there is. We will introduce a regular and practical method in the following part.

2. The method of constructing
If a problem has the following properties:

1). The function $f(x)$ is continuous in an interval, and its values at the endpoints of this interval meet a certain condition.

2). Its problem is to prove that there is a point ξ in this interval to meet an equation about $f(x)$ and $f'(x)$.

For example, the function $f(x)$ is continuous in interval $[a,b]$, and $f(a)=f(b)$, prove there is a point ξ, $\xi\in(a,b)$ to meet $f'(\xi)=f(\xi)$.

Then we can construct an auxiliary function as the following steps:

1). Changing the ξ of the result into x;

2). Transferring $f'(x)$ as $\frac{df(x)}{dx}$;

3). Arranging x and dx to be one side of the equation, and $f(x)$ and $df(x)$ to be another side;

4). Integrating in both sides meanwhile, and ignoring the constant c;

5). Arranging the equation to let its one side is 1, and the other side is a formula of x and $f(x)$, then this formula is the function which needs to be constructed.

3. Examples
Example 1: The function $f(x)$ is continuous in interval $[a,b]$, and $f(a)=f(b)$, prove there is a point ξ, $\xi\in(a,b)$ to meet $f'(\xi)=f(\xi)$.

The construction of the auxiliary function:

Now we have $f'(\xi)=f(\xi)$, so firstly we change its ξ in to x.

Then we get the equation $f'(x)=f(x)$.

That is $\frac{df(x)}{dx}=f(x)\ \therefore \frac{df(x)}{f(x)}=dx$

Integrate in both sides meanwhile, and ignoring the constant c.

That is $\int \frac{df(x)}{f(x)}=\int 1dx\ \therefore \ln f(x)=x\ \therefore f(x)=e^{x}$

The both sides of the equation is divided by e^{x} to make the right side to be 1.

Then $\frac{f(x)}{e^{x}}=1$, so we get the auxiliary function $g(x)=\frac{f(x)}{e^{x}}$.

Next we use this auxiliary function $g(x)=\frac{f(x)}{e^{x}}$ to prove the above problem.

Proof. Let $g(x)=\frac{f(x)}{e^{x}}$, $\because f(a)=f(b)=0$, $\therefore g(a)=g(b)=0$

According to Rolle's mean value theorem, then $\exists \xi \in (a,b)$ so that $g'(\xi)=0$.

That is $\frac{f'(\xi)e^{\xi}-f(\xi)e^{\xi}}{e^{2\xi}}=0$, $\therefore f'(\xi)-f(\xi)=0$, $\therefore f'(\xi)=f(\xi)$.

Example 2: The function $f(x)$ is continuous, and $f(a)=f(-a)$, prove there is ξ, $\xi \in (-a,0)\cup(0,a)$ so that $\xi f'(\xi)+2f(\xi)=0$.

The construction of the auxiliary function:

Now we have $\xi f'(\xi)+2f(\xi)=0$, so we firstly change its ξ into x.

Then we get the equation $xf'(x)+2f(x)=0$.

That is $\frac{xdf(x)}{dx}+2f(x)=0\ \therefore \frac{xdf(x)}{dx}=-2f(x)\ \therefore \frac{df(x)}{f(x)}=-2\frac{dx}{x}$.

Integrate in both sides meanwhile, and ignoring the constant c.

Then we get $\ln f(x)=-2\ln x$, $\therefore f(x)=x^{-2}$

The both sides of the equation is divided by x^{-2}, namely, times x^{2}.

$\therefore x^{2}f(x)=1$, so we get the auxiliary function $g(x)=x^{2}f(x)$

Next we use this auxiliary function $g(x)=x^{2}f(x)$ to prove the above problem.

Proof. Let $g(x)=x^2 f(x)$, $\because f(a)=f(-a)$, $\therefore g(a)=g(-a)$

According to Rolle's mean value theorem, then $\exists\xi\in(-a,a)$ so that $g'(\xi)=0$.

That is $2\xi f(\xi)+\xi^2 f'(\xi)=0$, $\xi\neq 0$, $\therefore 2f(\xi)+\xi f'(\xi)=0$

Example 3: The function $f(x)$ is continuous in the interval $(0,+\infty)$, and there are $0<a<b$ so that $f(a)=f(b)=1$, prove there is a point ξ, $\xi\in(a,b)$ to meet $2\xi f(\xi)\ln f(\xi)-f'(\xi)=0$.

The construction of the auxiliary function:
The result is seemed to be complicated, but the auxiliary function is easy to construct by following the method above.

Now we have $2\xi f(\xi)\ln f(\xi)-f'(\xi)=0$, so we firstly change its ξ into x.

Then we get the equation $2xf(x)\ln f(x)-f'(x)=0$

That is $2xf(x)\ln f(x)=\dfrac{df(x)}{dx}$, $\therefore \dfrac{df(x)}{f(x)\ln f(x)}=2xdx$

Integrate in both sides meanwhile, and ignoring the constant c,

That is $\displaystyle\int\frac{df(x)}{f(x)\ln f(x)}=\int 2xdx$, $\therefore \displaystyle\int\frac{d\ln f(x)}{\ln f(x)}=x^2$, $\therefore \ln(\ln f(x))=x^2$,

$\therefore \ln f(x)=e^{x^2}$, $\therefore \dfrac{\ln f(x)}{e^{x^2}}=1$, so we get the auxiliary function $g(x)=\dfrac{\ln f(x)}{e^{x^2}}$

Next we use this auxiliary function $g(x)=\dfrac{\ln f(x)}{e^{x^2}}$ to prove the above problem.

Proof. Let $g(x)=\dfrac{\ln f(x)}{e^{x^2}}$, $f(a)=f(b)=1$, $\therefore g(a)=g(b)=0$

According to Rolle's mean value theorem, then $\exists\xi\in(a,b)$ so that $g'(\xi)=0$

That is $\dfrac{\dfrac{f'(\xi)}{f(\xi)}e^{\xi^2}-2\xi\ln f(\xi)e^{\xi^2}}{e^{2\xi^2}}=0$, $\therefore \dfrac{f'(\xi)}{f(\xi)}-2\xi\ln f(\xi)=0$

$\therefore 2\xi f(\xi)\ln f(\xi)-f'(\xi)=0$

4. Reference

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