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A comparative study of numerical solutions of second order ordinary differential equations with boundary value problems by shooting method & finite difference method

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Abstract
This study aims at comparing the different time level schemes of different Method to obtain numerical solutions of second order ordinary differential equations. The numerical solutions are obtained for ordinary differential equation with boundary conditions for different values of x. Then the numerical solutions are compared with analytical solution for the considered problem. The comparison of numerical results with the analytical solution reveals that the Finite Difference Method gives the best numerical approximation to analytical solution. The error analysis of different method also discussed. This study can be used to suggest the better accuracy for solving second order ordinary differential equation. This comparative study provides better result for numerical solution of second order ordinary differential equation.

Keywords: Second order differential equation, numerical accuracy, shooting method, finite difference method

Introduction

It is well known that differential equations are in general the backbone of physical system. The physical systems are modeled usually either by ordinary differential or partial differential equation. Various exact and numerical method are available to solve different ordinary and partial differential equations. But in actual practice the variables and coefficients in the differential equation are not crisp. As those, are obtained by some experiment or experience. So, we need to solve ordinary and partial differential equations accordingly, that is interval ordinary and partial differential equations are to be solved [6]. There are many problems in the field of science, engineering and technology which can be solved by ordinary differential equations formulation. This research will compare the accuracy of various method Shooting Method and Finite Difference Method, in completing numerical solutions of second order ordinary differential equations, which is limited to certain boundary condition [2, 5].

Shooting Method for Boundary Value Problem

This method works in three stages

i) The given BVP is transferred in two IVPs,
ii) Solution of these two IVPs can be determined by Taylor's series or Runge-Kutta or any other method,
iii) Combination of these two solutions is the required solution of the given BVP.
Reduction to two IVPs [1].

Let the BVP be

$$y'' - p(x)y' - q(x)y = r(x) \text{ with } y(a) = \alpha \text{ and } y(b) = \beta \quad (1)$$

Suppose that $u(x)$ is the unique solution to the IVP

$$u''-p(x)u'(x)-q(x)u(x)=r(x) \text{ with } u(a)=\alpha \text{ and } u'(a)=0 \tag{2}$$

Furthermore, suppose that $u(x)$ is the unique solution to the IVP

$$v''-p(x)v'(x)-q(x)v(x)=0 \text{ with } v(a)=0 \text{ and } v'(a)=1 \tag{3}$$

Then the linear combination

$$y(x)=c_1u(x)+c_2v(x) \tag{4}$$

is the solution of equation 1.1 for some constant c_1, c_2

As $y(a)=\alpha$ and $y(b)=\beta$, $\alpha=c_1u(a)+c_2v(a)$ and $\beta=c_1u(b)+c_2v(b)$

From the first equation, $c_1=1$ and From second equation, $c_2=\dfrac{\beta-u(b)}{v(b)}$

Hence equation 1.4 reduces to

$$y(x)=u(x)+\frac{\beta-u(b)}{v(b)}v(x) \tag{5}$$ [7, 9]

This is the solution of the BVP 1.1 and its also satisfies the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$

Problem 1:
$y''=y-x$ with y(0) = 0, y(1) = 0.

The second order Runge-Kutta method is used to solve the initial value problem with h = 0.25
Here p(x) = 0, q(x) = 1, r(x) = -x, a = 0, b = 1, α = 0, β = 0.

Now two IVPs are

$u'=w$ with u(0) = 0

$$w'=u-x \text{ and with } w(0) = u'(0) = 0 \tag{6}$$

And

$v'=z$ with v(0) = 0

$$z'=v \text{ and with } z(0) = v'(0) = 1. \tag{7}$$

Solution of the system 1.6 is shown below.

i	**x_i**	**u_i**	**w_i**	**k_1**	**k_2**	**l_1**	**l_2**	**u_{i+1}**	**w_{i+1}**
0	0.0	0.00000	0.00000	0.00000	0.00000	0.00000	-0.04000	0.00000	-0.02000
1	0.2	0.00000	-0.02000	-0.00400	-0.01200	-0.04000	-0.08080	-0.00800	-0.08040
2	0.4	-0.00800	-0.08040	-0.01608	-0.03240	-0.08160	-0.12482	-0.03224	-0.18361
3	0.6	-0.03224	-0.18361	-0.03672	-0.06201	-0.12645	-0.17379	-0.08161	-0.33373
4	0.8	-0.08161	-0.33373	-0.06675	-0.10201	-0.17632	-0.22967	-0.16598	-0.53672

Solution of the system equation (7) is shown in the following table:

i	**x_i**	**v_i**	**z_i**	**k_1**	**k_2**	**l_1**	**l_2**	**v_{i+1}**	**z_{i+1}**
0	0.0	0.00000	1.00000	0.20000	0.20000	0.00000	0.04000	0.20000	1.02000
1	0.2	0.20000	1.02000	0.20400	0.21200	0.04000	0.08080	0.40800	1.08040
2	0.4	0.40800	1.08040	0.21608	0.23240	0.08160	0.12482	0.63224	1.18361
3	0.6	0.63224	1.18361	0.23672	0.26201	0.12645	0.17379	0.88161	1.33373
4	0.8	0.88161	1.33373	0.26675	0.30201	0.17632	0.22967	1.16598	1.53672

Now, $c = \frac{\beta - u(b)}{v(b)} = \frac{0 - u(b)}{v(1)} = \frac{0.16598}{1.16598} = 0.142352$

The value of y(x) given by $y(x) = u(x) + cv(x) = u(x) + 0.142352\ v(x)$ are listed below:

x	u(x)	v(x)	y(x)	y_{exact}
0.2	0.00000	0.20000	0.02847	0.02868
0.4	-0.00800	0.40800	0.05008	0.05048
0.6	-0.03224	0.63224	0.05776	0.05826
0.8	-0.08161	0.88161	0.04389	0.04429
1.0	-0.16598	1.16598	0.00000	0.00000

Finite Difference Method

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by an standard procedure. These roots are the values of the required solution at the pivotal points.

The working expressions for the central difference approximations to the first four derivatives of y_i are as under:

$$y_i^{'} = \frac{1}{2h}(y_{i+1} - y_{i-1})$$

$$y_i^{''} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$$y_i^{'''} = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2})$$

The accuracy of this method depends on the size of the sub interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases [10].

Problem 1

Finite difference method

We divide the interval (0, 1) into five sub-intervals so that h = 1/5 and the pivot points are at $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$

The differential equation is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = y_i - x_i$$

$$25y_{i+1} - 51y_i + 25y_{i-1} = -x_i$$

Using $y_0 = y_4 = 0$, we get the system of equations

$25y_2 – 51y_1 = -0.2$

$25y_3 – 51y_2 + 25y_1 = -0.4$

$25y_4 – 51y_3 + 25y_2 = -0.6$

$25y_3 – 51y_4 = -0.8$

$y_1 = 0.02859$

$y_2 = 0.05033$

$y_3 = 0.05808$

$y_4 = 0.04415$

Problem 2: $y'' = -y - x$ with y(0) = 0, y(1) = 0.

Shooting Method procedures we find the solution:

y(0) = 0.0000

y(0.25) = 0.0446

y(0.5) = 0.0708

y(0.75) = 0.0610

y(1) = 0.00000

Finite Difference Method we find the solution:

y(0) = 0.0000

y(0.25) = 0.0442

y(0.5) = 0.0701

y(0.75) = 0.06040

y(1) = 0.00000

Exact solution:

y(0) = 0.0000

y(0.25) = 0.0440

y(0.5) = 0.0697

y(0.75) = 0.0600

y(1) = 0.00000

Error Analysis

Problem 1: Relative Percentage Error Table

x_i	Finite Difference Method	Shooting Method
0.0	0.000000000	0.000000000
0.2	0.314685314	0.732217573
0.4	0.297147385	0.792393026
0.6	0.308959835	0.858221764
0.8	0.316098442	0.903138406
1.0	0.000000000	0.000000000

Problem 2: Relative Percentage Error Table

x_i	Finite Difference Method	Shooting Method
0.0	0.000000000	0.000000000
0.25	0.454545454	1.363636400
0.50	0.573888091	1.578192253
0.75	0.666666666	1.666666667
1.0	0.000000000	0.000000000

Conclusion

This research will compare the accuracy of various method Shooting Method and Finite Difference Method, in completing numerical solutions of second order ordinary differential equations, which is limited to certain boundary condition. This comparative study provides better result for numerical solution of second order ordinary equation.

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