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Sufficient conditions for time delay induced instability and oscillation

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Abstract
This paper provides some criteria to guarantee the existence of permanent oscillation for delayed differential equations which are induced by time delays. We present examples of simulations which confirm the correctness of our results.

Keywords: Delayed differential equation, permanent oscillation, delay

Introduction
Time-delay differential equations have long been used for mathematical modeling to model various phenomena in many practical applications, especially in engineering and biological sciences. For example, a lot of mechanical control systems are being implemented using communication networks in which time delays inevitably arise in the communication channels [1-3]. Due to the finite switching speed of the neuron amplifiers, and the presence of a multitude of parallel pathways with a variety of axon sizes and lengths, neural networks usually have discrete delays and distributed delays [4-7]. The dynamical behavior of predator-prey models depend on the past history of the system, many biological models have been incorporated time delays due to maturation time and capturing time [8-12]. The dynamic behaviors of various time delays differential equations have been discussed [13-24]. Many researchers have found that time delay may affect the dynamical performance and even induced oscillation [25-30]. However, a natural question is, under what conditions will instability, oscillation or chaos arise in system involving time delay? To the best of our knowledge, it is still an open problem. This paper from mathematical point of view provides some sufficient conditions under which time delay induced oscillation will occur for a delayed system.

Preliminaries
Consider the following delayed differential equation:

$$x'(t) = f(x(t), x_1(t-\tau_1), x_2(t-\tau_2), \ldots, x_n(t-\tau_n)), f(0) = 0, x \in R^n. \tag{1}$$

The initial condition is $x_i(t) = \varphi_i(t)$, $t \in [\overline{\tau}, 0]$, where $\overline{\tau}=\max\{\tau_1, \tau_2, \ldots, \tau_n\}$. The linearized form of equation (1) around the zero point is as follows:

$$x'(t) = Ax(t) + Bx(t-\tau) \tag{2}$$

Where $x(t)=(x_1(t), x_2(t), \ldots, x_n(t))^T$ $x(t-\tau)=(x_1(t-\tau_1), x_2(t-\tau_2), \ldots, x_n(t-\tau_n))^T$ $\tau_i \geq 0$ (i=1, 2, ..., n). Both $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ are n by n matrices.
Assume that x^* is an equilibrium point of equation (1). In order to discuss the stability of x^*, make the change of variable $y = x - x^*$ and we only need to deal with the stability of the zero equilibrium point. Thus, we assume that both equation (1) and system (2) have a unique trivial solution.
In this paper we adopt the following norms of vectors and matrices [31]: $|x(t)| = \sum_{i=1}^{n}|x_i(t)|$, $\|A\| = \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}|$, the measure μ (A)is defined by $\mu(A) = \lim_{\theta\to 0^+}\frac{|I+\theta A|-1}{\theta}$,

which for the chosen norms reduces to $\mu(A) = \max_{1\le j\le n}(a_{jj} + \sum_{i=1}^{n}|a_{ij}|)$.

Definition 1 Assume that the trivial solution of system (1) is Lyapunov stable but is not asymptotically stable or exponentially stable for delays $\tau_i \le \tau^*$ (i=1, 2, ..., n). Equation (1) generates an oscillatory solution when $\tau_i > \tau^*$ (i=1, 2, ..., n) which is called time delay induced oscillation.

Main Results

It is known that the trivial solution of equation (1) is unstable if and only if the trivial solution of system (2) is unstable. Therefore, in order to prove the instability of the trivial solution of equation (1), we can deal only with the instability of the trivial solution of system (2).

Theorem 1 Assume that equation (1) has a unique trivial solution which is Lyapunov stable, and all solutions of equation (1) are bounded. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ be characteristic roots of matrix *A* and *B*, respectively. If there exists some β_i $i \in \{1, 2, ..., n\}$ such that

$$0 < \beta_i, |\alpha_i| < \beta_i \; i \in \{1, 2, ..., n\}, \tag{3}$$

Or

$$0 < Re(\beta_i), |\,Re(\alpha_i)| < Re(\beta_i) \; i \in \{1, 2, ..., n\}. \tag{4}$$

Then there is time delay induced oscillation for equation (1).

Proof We only consider the instability of the trivial solution of system (2). The characteristic equation of system (2) is the following:

$$\det\left(\lambda I_{ij} - A - Be^{-\lambda\tau_{ij}}\right) = 0 \tag{5}$$

where *I* is the identity matrix. Since $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ are the eigenvalues of *A* and *B* respectively, we have immediately that

$$\prod_{j=1}^{n}\left(\lambda - \alpha_j - \beta_j e^{-\lambda\tau_j}\right) = 0 \tag{6}$$

So, we are led to an investigation of the nature of some $i \in \{1, 2, ..., n\}$ such that

$$\lambda - \alpha_i - \beta_i e^{-\lambda\tau_i} = 0 \tag{7}$$

Equation (7) is a transcendental equation. Generally speaking we cannot find all solutions of this equation. However, we prove that there exists a positive eigenvalue of equation (7) under the restrictive condition (3) or (4). Indeed, let $\phi(\lambda) = \lambda - \alpha_i - \beta_i e^{-\lambda\tau_i}$, then $\phi(\lambda)$ is a continuous function of λ. Obviously, if the condition (3) holds, then $\phi(0) = -\alpha_i - \beta_i \le |\alpha_i| - \beta_i < 0$. Noting that $\tau_i \ge 0$, and $\lim_{\lambda\to+\infty} e^{-\lambda\tau_i} = 0$, therefore, there exists $\tilde{\lambda}$ (> 0) such that $\phi(\tilde{\lambda}) = \tilde{\lambda} - \alpha_i - \beta_i e^{-\tilde{\lambda}\tau_i} > 0$. According to the well known Intermediate Value Theorem, there exists a positive value of λ sayλ_1, $\lambda_1 \in (0, \tilde{\lambda})$ such that $\phi(\lambda_1) = \lambda_1 - \alpha_i - \beta_i e^{-\lambda_1\tau_i} = 0$. In other words, equation (7) has a positive characteristic root. If the condition (4) holds, then let $\lambda = \sigma + i\omega$, $\alpha_i = \alpha_{i1} + i\alpha_{i2}$, $\beta_i = \beta_{i1} + i\beta_{i2}$. Separating the real and imaginary parts from equation (7), we have

$$\sigma = \alpha_{i1} + \beta_{i1}e^{-\sigma\tau_i}\cos(\omega\tau_i) + \beta_{i2}e^{-\sigma\tau_i}\sin(\omega\tau_i) \tag{8}$$

$$\omega = \alpha_{i2} - \beta_{i1}e^{-\sigma\tau_i}\sin(\omega\tau_i) + \beta_{i2}e^{-\sigma\tau_i}\cos(\omega\tau_i) \tag{9}$$

We show that equation (8) has a positive real root. Let

$$\varphi(\sigma) = \sigma - \alpha_{i1} - \beta_{i1}e^{-\sigma\tau_i}\cos(\omega\tau_i) - \beta_{i2}e^{-\sigma\tau_i}\sin(\omega\tau_i) \tag{10}$$

Obviously, $\varphi(\sigma)$ is also a continuous function of σ. When σ = 0 we have $\varphi(0) = -\alpha_{i1} - \beta_{i1}\cos(\omega\tau_i) - \beta_{i2}\sin(\omega\tau_i)$. If τ_i is suitably small, we have $\sin(\omega\tau_i) \to 0$, $\cos(\omega\tau_i) \to 1$, and $\varphi(0) = -\alpha_{i1} - \beta_{i1}\cos(\omega\tau_i) - \beta_{i2}\sin(\omega\tau_i) \le |Re(\alpha_i)| - Re(\beta_i) < 0$. Also $\lim_{\sigma\to+\infty} e^{-\sigma\tau_i} = 0$, thus there exists a suitably large $\tilde{\sigma}$ (> 0) such that $\varphi(\tilde{\sigma}) = \tilde{\sigma} - \alpha_{i1} - \beta_{i1}e^{-\tilde{\sigma}\tau_i}\cos(\omega\tau_i) - \beta_{i2}e^{-\tilde{\sigma}\tau_i}\sin(\omega\tau_i) > 0$. Again by means of the Intermediate Value Theorem, there exists a $\sigma_1 \in (0, \tilde{\sigma})$ such that $\varphi(\sigma_1) = \sigma_1 - \alpha_{i1} - \beta_{i1}e^{-\sigma_1\tau_i}\cos(\omega\tau_i) - \beta_{i2}e^{-\sigma_1\tau_i}\sin(\omega\tau_i) = 0$. This means that the characteristic value λ has a positive real part when τ_i is suitably small. In a time delayed system, it was emphasized that if the trivial solution is unstable for small time delay, then the instability of trivial solution will maintain as delay increases. Therefore, the trivial solution of system (2) is unstable for any time delays, implying that the trivial solution of system (1) is unstable. Noting that all solutions of equation (1) are bounded and hence system (1) generates an oscillatory solution.

Theorem 2 Assume that equation (1) has a unique trivial solution which is Lyapunov stable, and all solutions of equation (1) are bounded. If there exists $\tau > \tau^*$ such that

$$\mu(A) + \|B\| > 0 \tag{11}$$

holds, then there is time delay induced oscillation for equation (1).

Proof For avoiding unnecessary complexity, we consider each $\tau_i = \tau$ (i=1, 2, ..., n) in system (2). Let us rewrite system (2) in component form

$$x_i'(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t-\tau) \ , i = 1,2,\dots,n. \tag{12}$$

Thus, we have

$$|x_i'(t)| \le a_{ii} x_i(t) + \sum_{i\neq j, j=1}^{n} |a_{ij}||x_j(t)| + \sum_{j=1}^{n} |b_{ij}||x_j(t-\tau)| \ , i = 1,2,\dots,n \tag{13}$$

Let $y(t) = \sum_{i=1}^{n} |x_i(t)|$, we have

$$y'(t) \le \mu(A) y(t) + \|B\| y(t-\tau) \tag{14}$$

Consider the scalar delayed differential equation

$$z'(t) = \mu(A) z(t) + \|B\| z(t-\tau) \tag{15}$$

According to the comparison theorem of differential equation, we have $y(t) \le z(t)$. We show that the solution of (15) is unstable. Suppose this is not the case, then the characteristic equation associated with (15) given by

$$\lambda = \mu(A) + \|B\| e^{-\lambda\tau} \tag{16}$$

does not have any positive roots. From (16) we have

$$\lambda - \mu(A) - \|B\| e^{-\lambda\tau} = 0 \tag{17}$$

Equation (17) is a transcendental equation. We prove that there exists a positive characteristic
root of equation (17) under the restrictive condition (11). Let $f(\lambda) = \lambda - \mu(A) - \|B\| e^{-\lambda\tau}$, then $f(\lambda)$ is a continuous function of λ and $f(0) = -\mu(A) - \|B\| = -(\mu(A) + \|B\|) < 0$, since $\mu(A) + \|B\| > 0$. Obviously, there exists a suitably large λ^* such that $f(\lambda^*) = \lambda^* - \mu(A) - \|B\| e^{-\lambda^* \tau} > 0$. Again by the Intermediate Value Theorem, there exists a positive value of λ say λ_0, $\lambda_0 \in (0, \lambda^*)$ such that Let $f(\lambda_0) = \lambda_0 - \mu(A) - \|B\| e^{-\lambda_0 \tau} = 0$. Thus (17) has a positive characteristic root. Therefore, the trivial solution of system (2) is unstable, implying that the trivial solution of equation (1) is unstable, and equation (1) generates an oscillatory solution since all the solutions of equation (1) are bounded.

Theorem 3 Assume that equation (1) has a unique trivial solution which is Lyapunov stable, and all solutions of equation (1) are bounded. If A=diag$(a_{ii})_{n\times n}$ and the following conditions for some $i \in \{1, 2, ..., n\}$ hold

$$\det(A+B) \neq 0, \tag{18}$$

$$\exp(-a_{ii}\tau_i)|b_{ii}| \, e\tau_i > 1 + \sum_{j=1, j\neq i}^{n} |b_{ij}| \exp(-a_{jj}\tau_j) \exp((a_{jj} - a_{ii}) t) e\tau_j. \tag{19}$$

Then there is time delay induced oscillation for equation (1).

Proof According to the basic knowledge of linear algebra, condition (18) implies that system (2) has a unique trivial solution. We shall prove that system (2) has an oscillatory solution. Since A=diag$(a_{ii})_{n\times n}$, the component form of (2) is the following

$$x_i'(t) = a_{ii} x_i(t) + \sum_{j=1}^{n} b_{ij} x_j(t-\tau_j) \ , i = 1,2,\dots,n. \tag{20}$$

Let $x_i(t) = \exp(a_{ii}t)\, y_i(t)$, then $x_i'(t) = a_{ii} \exp(a_{ii}t)\, y_i(t) + a_{ii} \exp(a_{ii}t)\, y'_i(t) = a_{ii} x_i(t) + a_{ii} \exp(a_{ii}t)\, y'_i(t)$. System (20) changes to the following

$$y'_i(t) = \exp(-a_{ii}t) \sum_{j=1}^{n} b_{ij} \exp(a_{jj}(t-\tau_j)) y_j(t-\tau_j), i = 1,2,\dots,n. \tag{21}$$

Let $c_{ij} = \exp(-a_{ii}t)\, b_{ij} \exp\left(a_{jj}(t-\tau_j)\right) = \exp\left((a_{jj} - a_{ii})t\right) b_{ij} \exp\left(-a_{jj}\tau_j\right)$, then (21) changes as the follows:

$$y'_i(t) = \sum_{j=1}^{n} c_{ij} y_j(t-\tau_j), i = 1,2,\dots,n. \tag{22}$$

The characteristic equation corresponding to system (22) is

$$\det\left(\lambda I_{ij} - c_{ij}e^{-\lambda\tau_j}\right) = 0 \tag{23}$$

where $\mathrm{I} = I_{ij}$ is an identity n by n matrix. If system (2) does not have an oscillatory solution, then characteristic equation (23) must have a real negative root say ρ such that

$$\det\left(\rho I_{ij} - c_{ij}e^{-\rho\tau_j}\right) = 0 \tag{24}$$

By the Gerschgorin's circle theorem [32], ρ satisfies

$$|\rho - c_{ii}e^{-\rho\tau_i}| \le \sum_{j=1,j\ne i}^{n} |c_{ij}|\, e^{-\rho\tau_j} \tag{25}$$

for some $i \in \{1, 2, ..., n\}$. From (25) we have

$$|\rho| = |c_{ii}e^{-\rho\tau_i} + \rho - c_{ii}e^{-\rho\tau_i}| \ge |c_{ii}e^{-\rho\tau_i}| - |\rho - c_{ii}e^{-\rho\tau_i}| \ge |c_{ii}e^{-\rho\tau_i}| - \sum_{j=1,j\ne i}^{n} |c_{ij}|\, e^{-\rho\tau_j} \tag{26}$$

Since $\rho < 0$, then $-\rho = |\rho|$. From (26) we have

$$|\rho| + \sum_{j=1,j\ne i}^{n} |c_{ij}|\, e^{|\rho|\tau_j} \ge |c_{ii}| e^{|\rho|\tau_i} \tag{27}$$

Thus,

$$1 + \sum_{j=1,j\ne i}^{n} |c_{ij}| \frac{e^{|\rho|\tau_j}}{|\rho|} = 1 + \sum_{j=1,j\ne i}^{n} |c_{ij}| \frac{\tau_j\, e^{|\rho|\tau_j}}{|\rho|\tau_j} \ge |c_{ii}| \frac{e^{|\rho|\tau_i}}{|\rho|} = |c_{ii}| \frac{\tau_i\, e^{|\rho|\tau_i}}{|\rho|\tau_i} \tag{28}$$

Using inequality $e^x \ge xe$, for some $i \in \{1, 2, ..., n\}$ we have

$$1 + \sum_{j=1,j\ne i}^{n} |c_{ij}|\, e\tau_j \ge |c_{ii}| e\tau_i \tag{29}$$

Noting that $|c_{ii}| = \exp(-a_{ii}\tau_i)|b_{ii}|$. This means that (29) contradicts (19), implying that the trivial solution of system (2) is unstable, and equation (1) generates an oscillatory solution since all the solutions of equation (1) are bounded.

Simulation result

First we consider the following coupled Rayleigh-Duffing oscillators with delays [33]:

$$\begin{cases} x_1''(t) = a_1x_1(t) - b_1x_1^3(t) + c_1x_1'(t) - d_1[x_1'(t)]^3 + k_1[x_1'(t-\sigma_1) - x_2'(t-\sigma_2)] \\ \qquad + r_1[x_1(t-\mu_1) - x_2(t-\mu_2)] \\ x_2''(t) = a_2x_2(t) - b_2x_2^3(t) + c_2x_2'(t) - d_2[x_2'(t)]^3 + k_2[x_2'(t-\sigma_2) - x_1'(t-\sigma_1)] \\ \qquad + r_2[x_2(t-\mu_2) - x_1(t-\mu_1)] \end{cases} \tag{30}$$

It is convenient to write (30) as an equivalent four dimensional first-order system:

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = a_1x_1(t) - b_1x_1^3(t) + c_1x_2(t) - d_1[x_2(t)]^3 + k_1[x_2(t-\tau_2) - x_4(t-\tau_4)] \\ \qquad + r_1[x_1(t-\tau_1) - x_3(t-\tau_3)] \\ x_3'(t) = x_4(t) \\ x_4'(t) = a_2x_3(t) - b_2x_3^3(t) + c_2x_4(t) - d_2[x_4(t)]^3 + k_2[x_4(t-\tau_4) - x_2(t-\tau_2)] \\ \qquad + r_2[x_3(t-\tau_3) - x_1(t-\tau_1)] \end{cases} \tag{31}$$

where $\tau_1 = \mu_1, \tau_2 = \sigma_1, \tau_3 = \mu_2, \tau_4 = \sigma_2$. The linearized system of (31) around x=0 is the following:

$$\mathrm{X}'(\mathrm{t}) = A_1X(t) + B_1X(t-\tau) \tag{32}$$

where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_2 & c_2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1 & k_1 & -r_1 & -k_1 \\ 0 & 0 & 0 & 0 \\ -r_2 & -k_2 & r_2 & k_2 \end{bmatrix}$. In system (31) we first set a_1=0.05, a_2=0.15, b_1=0.18, b_2=0.22, c_1=-2.55, c_2=-2.65, d_1=0.15, d_2=0.16, k_1=1.55, k_2=1.75, r_1=2.48, r_2=2.56, then the eigenvalues of matrix A_1 are 0.0195, -2.5695, 0.0554, -2.7054, and B_1 has an eigenvalue 3.3000. The trivial solution of system (31) is convergent as delays are [2.7, 2.8, 3.3, 3.5] (see Fig. 1(a)). Since |-2.7054|<3.3000, when we increased time delays as [3.5, 3.8, 3.6, 3.8], oscillatory solution appeared based on Theorem 1 (see Fig. 1(b)).

Then we consider a mechanical controlled system with time delays [34]:

$$\begin{cases} m_1 x_1''(t) + c_1 x_1'(t) + k_1 x_1(t) = g_1 x_2'(t-\sigma_2) + n_1 x_1(t-\mu_1) \\ m_2 x_2''(t) + c_2 x_2'(t) + k_2 x_2(t) = g_2 x_1'(t-\sigma_1) + n_2 x_2(t-\mu_2) \end{cases} \quad (33)$$

(33) can be written as an equivalent four dimensional first-order system:

$$\begin{cases} x_1'(t) = \frac{1}{m_1} x_2(t) \\ x_2'(t) = -c_1 x_2(t) - k_1 x_1(t) + g_1 x_4(t-\tau_4) + n_1 x_1(t-\tau_1) \\ x_3'(t) = \frac{1}{m_2} x_4(t) \\ x_4'(t) = -c_2 x_4(t) - k_2 x_3(t) + g_2 x_2(t-\tau_2) + n_2 x_3(t-\tau_3) \end{cases} \quad (34)$$

The matrix form of system (34) is as follows:

$$X'(t) = A_2 X(t) + B_2 X(t-\tau) \quad (35)$$

where $A_2 = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & g_1 \\ 0 & 0 & 0 & 0 \\ 0 & g_2 & n_2 & 0 \end{bmatrix}$. $a_{12} = \frac{1}{m_1}$, $a_{21} = -k_1$, $a_{22} = -c_1$, $a_{34} = \frac{1}{m_2}$, $a_{43} = -k_2$, $a_{44} = -c_2$.

In system (34) we set m_1=2.5, m_2=5.0, c_1 =0.64, c_2 =0.75, k_1 =0.65, k_2 =0.55, g_1 =0.45, g_2 =0.75, n_1 =0.72, $n_2 = -0.35$. Then $\mu(A_2) = 0.65$, $\|B\| = 0.75$, and $\mu(A_2) + \|B\| = 1.4 > 0$. The trivial solution of system (34) is convergent as delays are selected as [0.25, 0.26, 0.27, 0.28] (see Fig. 2(a)). However, when we increased time delays as [1.65, 1.66, 1.67, 1.68], system (34) generates an oscillatory solution based on Theorem 2 (see Fig. 2(d)).

Finally, consider a five-node network model as follows:

$$\begin{cases} x_1'(t) = a_{11} x_1(t) + b_{11} f(x_1) + b_{12} f(x_2) + b_{13} f(x_3) + b_{14} f(x_4) + b_{15} f(x_5) \\ x_2'(t) = a_{22} x_2(t) + b_{21} f(x_1) + b_{22} f(x_2) + b_{23} f(x_3) + b_{24} f(x_4) + b_{25} f(x_5) \\ x_3'(t) = a_{33} x_3(t) + b_{31} f(x_1) + b_{32} f(x_2) + b_{33} f(x_3) + b_{34} f(x_4) + b_{35} f(x_5) \\ x_4'(t) = a_{44} x_4(t) + b_{41} f(x_1) + b_{42} f(x_2) + b_{43} f(x_3) + b_{44} f(x_4) + b_{45} f(x_5) \\ x_5'(t) = a_{55} x_5(t) + b_{51} f(x_1) + b_{52} f(x_2) + b_{53} f(x_3) + b_{54} f(x_4) + b_{55} f(x_5) \end{cases} \quad (36)$$

where the activation function $f(x_i) = \tanh\big(x_i(t-\tau_i)\big)$, $i = 1,2,\dots,5$. The linearized system of (36) around the zero point is as follows:

$$X'(t) = A_3 X(t) + B_3 X(t-\tau) \quad (37)$$

where A_3=diag (a_{11}, a_{22}, a_{33}, a_{44}, a_{55}) is a diagonal matrix. Noting that $f'(x_i)|_{x_i=0} = \tanh'\big(x_i(t-\tau_i)\big)|_{x_i=0} = 1 - \tanh^2(0) = 1$, then $B_3 = (b_{ij})_{5\times 5}$. Set $a_{11} = -0.065$, $a_{22} = -0.085$, $a_{33} = -0.096$, $a_{44} = -0.075$, $a_{55} = -0.082$; $b_{11} = -2.85$, $b_{12} = -0.25$, $b_{13} = -0.45$, $b_{14} = -0.35$, $b_{15} = -0.55$; b_{21} =1.15, $b_{22} = -0.35$, $b_{23} = -0.25$, $b_{24} = -0.35$, $b_{25} = -0.25$; b_{31} =2.45, b_{32} =0.15, $b_{33} = -1.35$, b_{34} =0.35, $b_{35} = -0.25$; b_{41} =2.25, $b_{42} = -0.45$, b_{43} =0.20, $b_{44} = -1.35$, $b_{45} = -0.25$; b_{51} =1.65, $b_{52} = -0.24$, b_{53} =0.15, b_{54} =0.65, b_{55} =0.18. When time delays are selected as [0.38, 0.35, 0.36, 0.28, 0.35], we see the solution is convergent (Fig. 3(a)). However, when we increased delays as [0.45, 0.48, 0.46, 0.45, 0.47], system (36) generated an oscillatory solution based on Theorem 3 (see Fig. 3(d)). In this case we select i=1, and $\exp(-a_{11}\tau_1)|b_{11}|\,e\tau_1 = \exp(0.065 \times 0.45) \times 2.85 \times e \times 0.45 = 4.7055$. Noting that $\exp((a_{jj} - a_{11})t) \le 1$ for any $t \ge 0$ ($j = 2,3,4,5$), thus $1 + \sum_{j=2}^{5} |b_{1j}| \exp(-a_{jj}\tau_j) \exp((a_{jj} - a_{11})\,t) e\tau_j \le 1 + \sum_{j=2}^{5} |b_{1j}| \exp(-a_{jj}\tau_j)\, e\tau_j = 1 + 0.4866 + 0.6131 + 0.5820 + 0.7303 = 3.4120$. Obviously, $3.4120 < 4.7055$ and (19) holds.

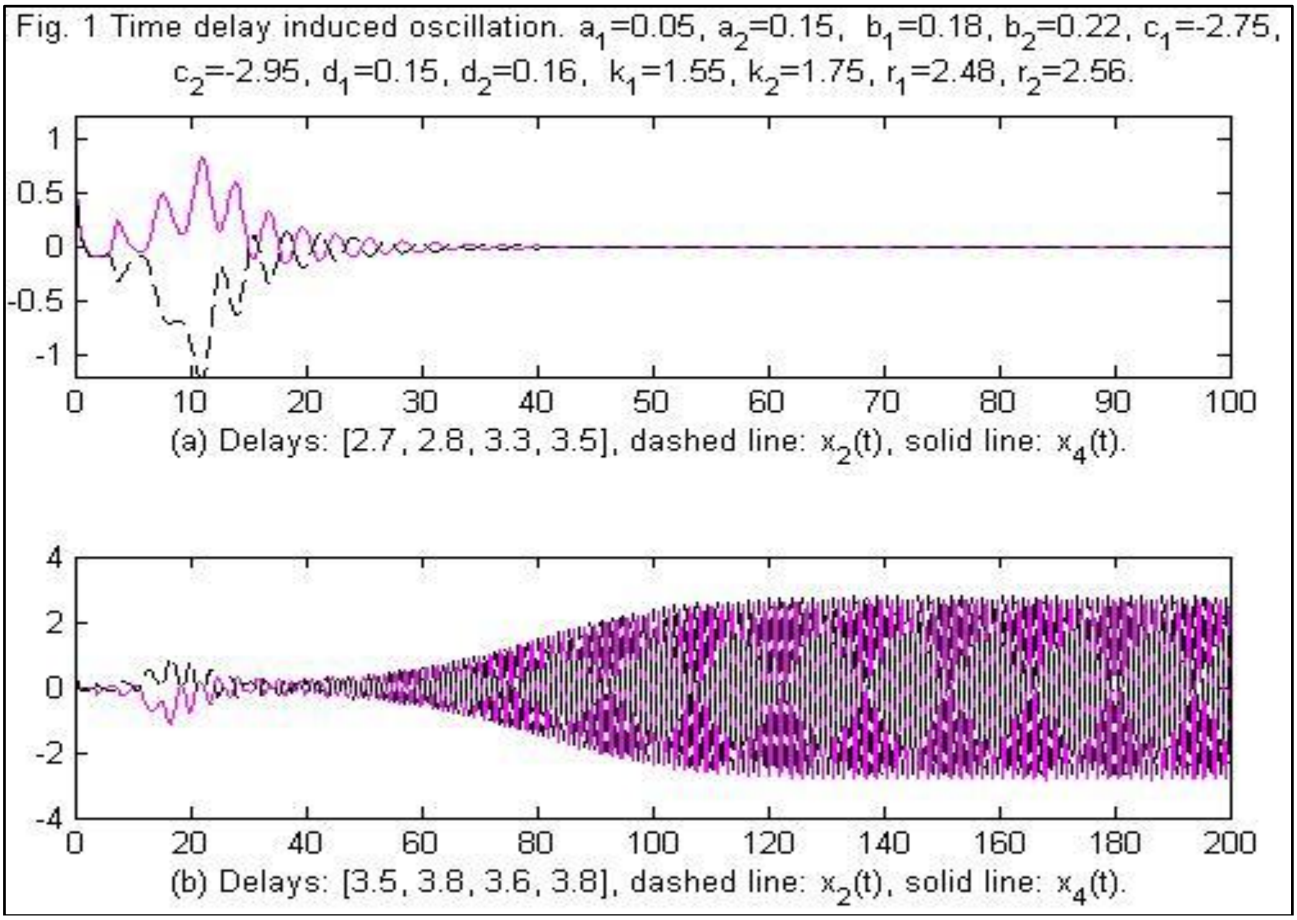


Fig. 1 Time delay induced oscillation. a_1=0.05, a_2=0.15, b_1=0.18, b_2=0.22, c_1=-2.75, c_2=-2.95, d_1=0.15, d_2=0.16, k_1=1.55, k_2=1.75, r_1=2.48, r_2=2.56.

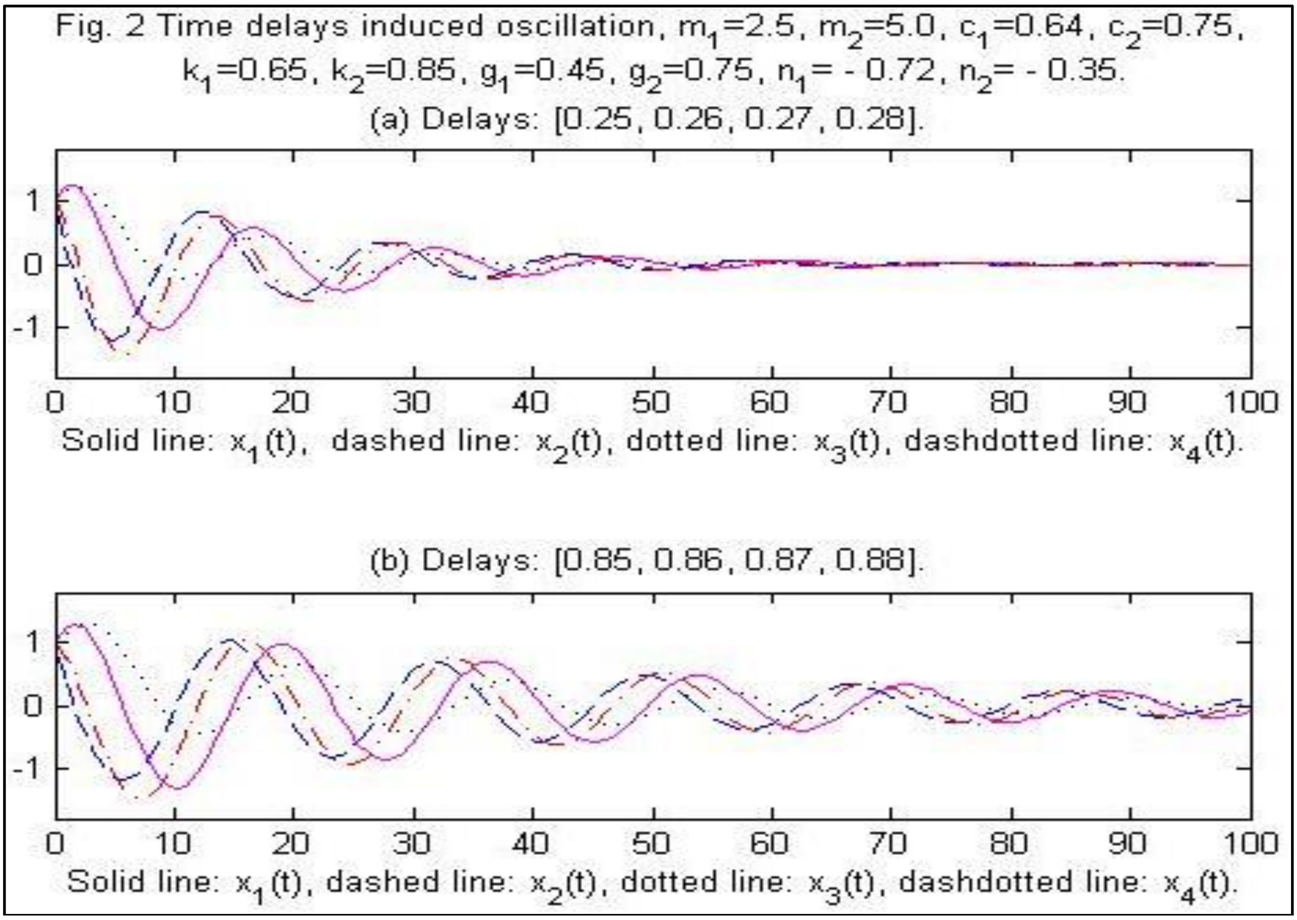
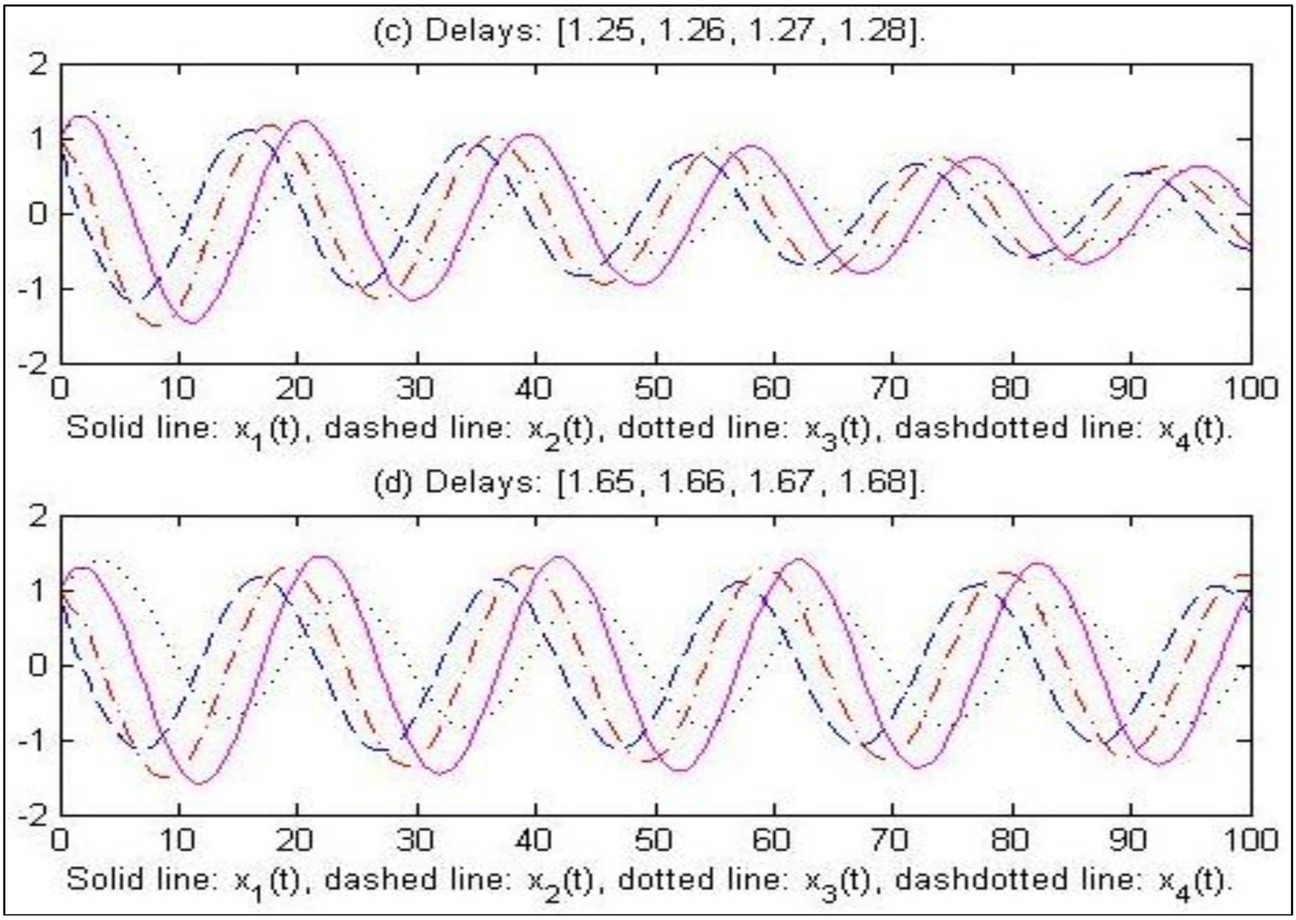


Fig. 2 Time delays induced oscillation, m_1=2.5, m_2=5.0, c_1=0.64, c_2=0.75, k_1=0.65, k_2=0.85, g_1=0.45, g_2=0.75, n_1= - 0.72, n_2= - 0.35.



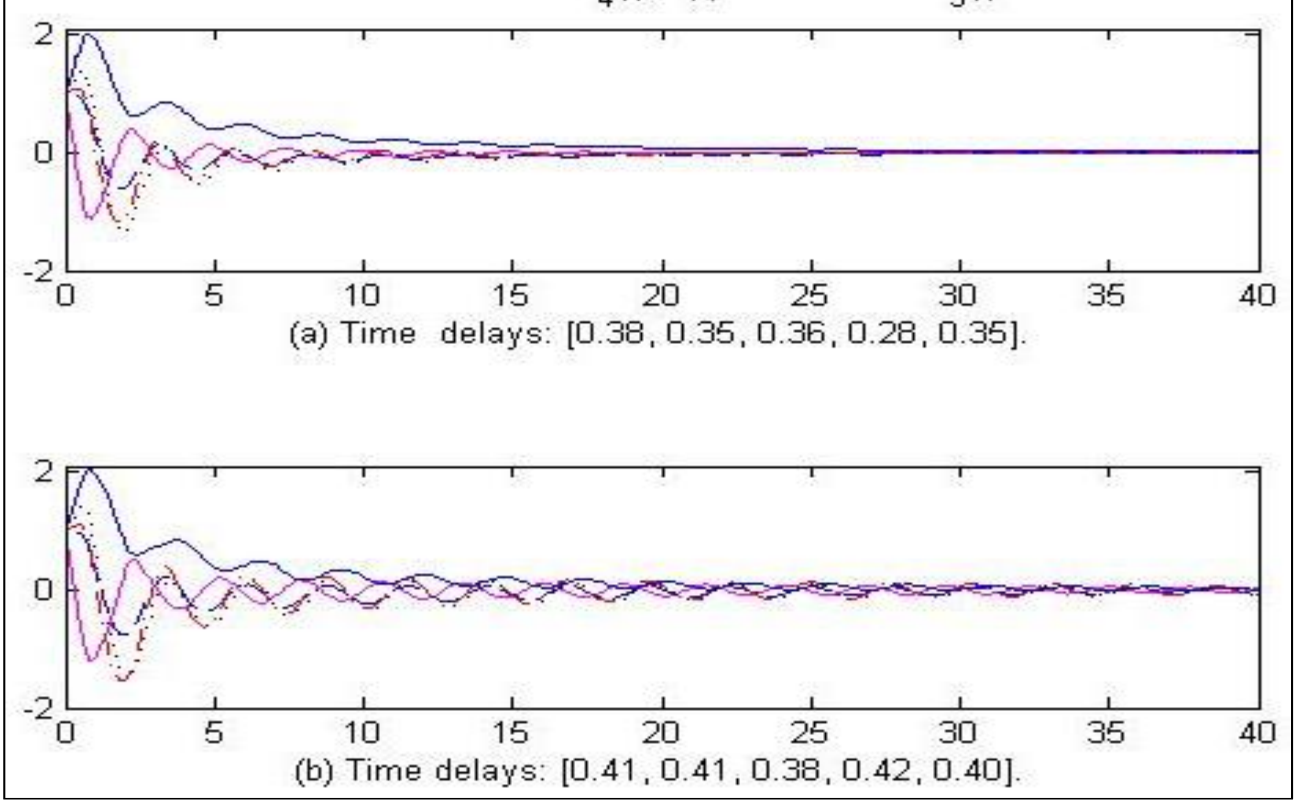
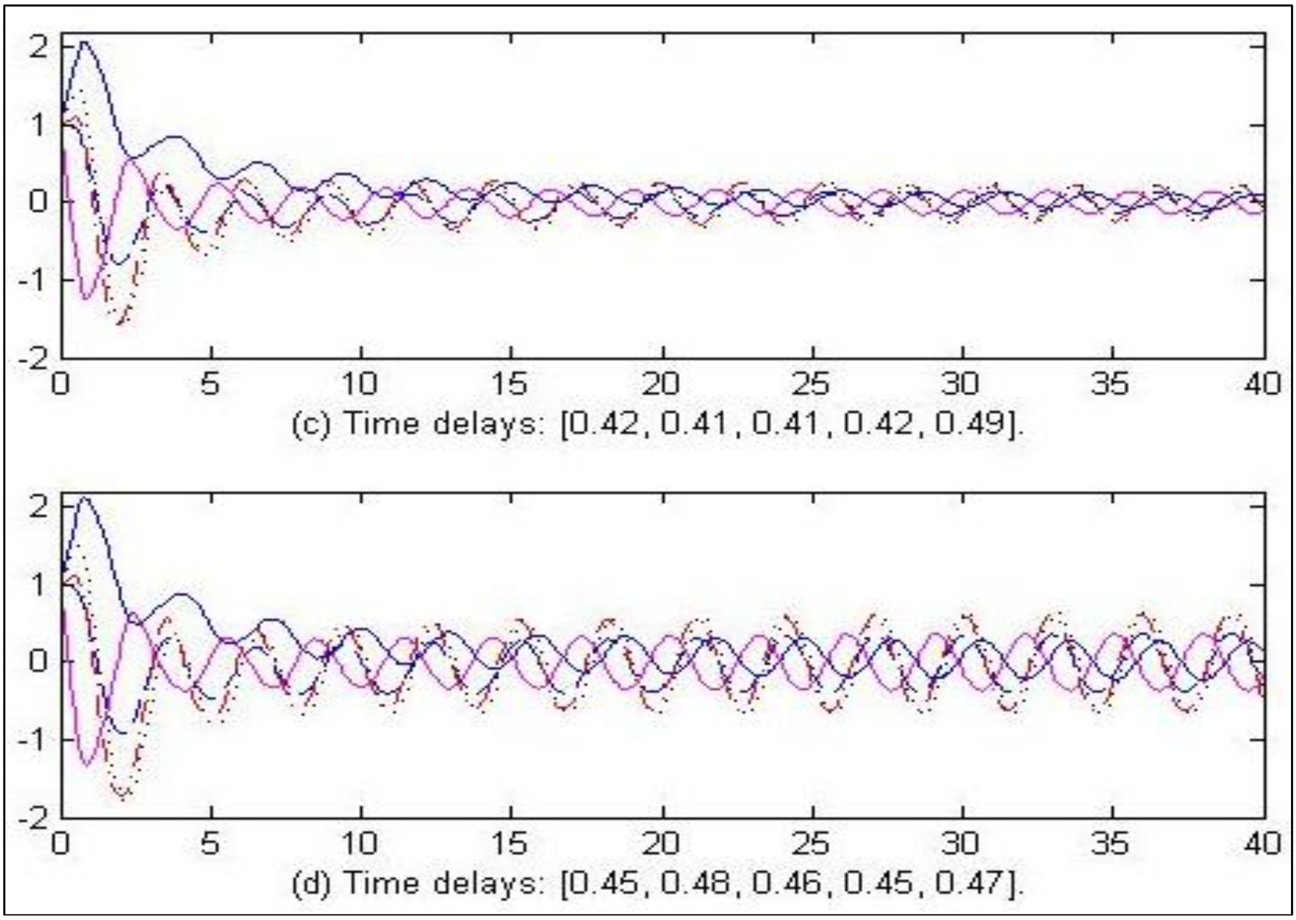


Fig. 3 Delay induced oscillation for different delay values (a)-(d). Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$, dashdotted line: $x_4(t)$, upper solid line: $x_5(t)$.



(c) Time delays: [0.42, 0.41, 0.41, 0.42, 0.49].

(d) Time delays: [0.45, 0.48, 0.46, 0.45, 0.47].

Conclusion

This paper considers the problem of time delay induced instability and oscillation. Some criteria have been given to ensure that there is time delay induced oscillation in delayed differential equations. Our simple criterion is easily to check. Figure 2 and 3 show that, as the delays increase the solutions of the systems change from stability to vibration, respectively. The simulation suggests that our criteria are only sufficient conditions.

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