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Multifractal analysis on the historic set in non-uniformly expanding systems

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Abstract
Authors conduct the multifractal analysis in a class of non-uniformly expanding systems. Combining methods of constructing Moran sets and assembling n-level bernoulli measures, they prove that the historic set of asymptotically additive potential carries full Hausdorff dimension unless it is empty set.

Keywords: Historic set, Moran set, non-uniformly expanding systems

1. Introduction
This paper is devoted to the study of the historic set of asymptotically additive potential in a class of one dimensional non-uniformly hyperbolic systems. Before formulating our results, we first present some background and motivation.

1.1. Historic set
We call (X,T) a topological dynamical system (for short TDS) if X is a compact metric space and $T: X \to X$ is a continuous map. We denote the set of continuous functions from X to $\mathbb{R}$ by $C(X, \mathbb{R})$. For any given $f \in C(X, \mathbb{R})$, let $S_n f := \sum_{i=0}^{n-1} f \circ T^i$ and $A_n f := \frac{1}{n} S_n f$ be the Birkhoff sum and the Birkhoff average of f respectively. We call the orbit $\{x, Tx, \cdots, T^n x, \cdots\}$ of x having "historical behavior" if the limit $\lim_{n\to\infty} A_n f(x)$ does not exists for some $f \in C(X, \mathbb{R})$. David. Ruelle firstly introduced this term in [18]. To pay tribute to Ruelle, we call a point $x \in X$ historic point if the orbit of x has historic behavior and call the set of all historic points as historic set. Flois. Takens discuss historic set further more in [19] and proposed the following problem: whether there are persistent classes of smooth dynamical systems such that the set initial states which give rise to orbits with historic behavior has positive lebesgue measure. This problem is called "Takens last problem", see [12, 13, 14, 5].
We call poin $x \in X$ "generic point" if the limit $\lim_{n\to\infty} A_n \varphi x$ exists for each $\varphi \in C(X, \mathbb{R})$. Limit $\lim_{n\to\infty} A_n \varphi x$ does not exists means that as time n goes to ∞, "$T^n x$" keeps having new ideas about what it want to do. The behavior of the orbit of "generic point" is predicable. However, the orbit of "historic point" contains more information, it has a history. Ruelle and Takens suggest that "historic set" is more important. For any given $f \in C(X, \mathbb{R})$, Let $\mathrm{H}(f; X, T) := \{x \in X : \lim_{n\to\infty} A_n f(x) \text{ does not exist}\}$ be the historic set of f. Then $\mathrm{H}(f; X, T)$ is a subset of historic set and historic set is the union of $\mathrm{H}(f; X, T)$ for all continuous function $f \in C(X, \mathbb{R})$.

Recently, the structure of historic set $\mathrm{H}(f;X,T)$ attracts great interests, see [1, 2, 3, 4, 6, 7, 8, 9, 11, 16, 17, 18, 19, 20]. By Birkhoff ergodic theory, we have $\mu(\mathrm{H}(f;X,T))=0$ for any given T-invariant measure μ. This implies that historic set is negligible in measure category. However, results of [1] overthrow this viewpoint. In most of the systems, especially uniformly hyperbolic systems, historic set carries full topological entropy and has full Hausdorff dimension. These means that historic set carries most of the information on the system. In finite subshifts and conformal repellers [1, 6, 8], authors prove that historic set has full Hausdorff dimension unless it is empty set. In systems with some mild specification property [20], Daniel. Tompson shows that either historic set carries full topological pressure or it is nothing. In [3], authors show that historic set is residual if it is not empty set. In systems with specification property [4], the same authors get a variational principal on the topological entropy of historic set.

In this paper, we consider the historic set of asymptotically additive function sequence in a class of non-uniformly expanding systems. We show that the Hausdorff dimension of the historic set in this systems has "dichotomy", i.e. it carries full Hausdorff dimension unless it is empty set. In the following, we introduce the non-uniformly expanding systems considered in this paper.

1.2. Non-uniformly expanding systems

We consider the pseudo-Markov maps. Let $\{I_i : i=1,\cdots,m\}$ be the subintervals of the unit interval [0, 1] and $T:\bigcup_{i=1}^{m} I_i \to$[0, 1] be a piecewise $C^{1+\gamma}$ map with exponent $\gamma>0$. Furthermore, we impose the following assumptions:

1. $\mathrm{int}(I_i)\bigcap \mathrm{int}(I_j)=\varnothing$ for $i\neq j$, $\mathrm{int}(I_i)$ is the interior of I_i,
2. For $i=1,\cdots,m$, $T|_{I_i}: I_i \to I$ is a $C^{1+\gamma}$ continuous surjective map and there is a unique $x_i\in I_i$ with $T(x_i)=x_i$ and $|T'(x_i)|\geq 1$,
3. $|T'(x)|>1$ for $x\notin\{x_1,x_2,\cdots,x_m\}$.

We call the point $x_i\in I_i$ parabolic fixed point if $T(x_i)=x_i$ and $|T'(x_i)|=1$. The above systems are called non-uniformly expanding systems. Parabolic fixed points make the non-uniformly expanding systems distinct from uniformly hyperbolic systems. This class of non-uniformly expanding maps contains the famous Manneville-Pomeau map, that is $T:[0,1]\to[0,1]$ defined by $T(x)=x+x^{1+\beta}(\mathrm{mod}1)$, where $0<\beta<1$.

The following set

$$\Lambda=\{x\in\bigcup_{i=1}^{m} I_i\colon T^n(x)\in[0,1],\ \forall\ n\geq 0\}$$

Is called the attractor of T. Obviously, Λ is T-invariant.

There is a natural symbolic coding map for the attractor of non-uniformly expanding systems. For $i=1,\ 2,\ ...,m$, T_i is the inverse of $T|_{I_i}\colon\ I_i\to$[0,1]. Let $A=\{1,2,\cdots,m\}$ be the alphabet, $\Sigma=A^{\mathbb{N}}$ be the full shift over m symbols and $\sigma:\Sigma\to\Sigma$ be the left shift with $\sigma((\omega_n)_{n\geq 1})=(\omega_n)_{n\geq 2}$. We define the coding map $\Pi:\Sigma\to$[0, 1] as

$$\Pi(\omega):=\lim_{n\to\infty} T_{\omega_1}\circ T_{\omega_2}\cdots\circ T_{\omega_n}([0,1]).$$

It is seen that $\Pi\Sigma=\Lambda$ and $\Pi\circ\sigma(\omega)=T\circ\Pi(\omega)$. We point out coding map Π is a bijection except for at most countably many points.

1.3. Asymptotically additive potential

Let $\Phi=\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on X. By convention of Feng, see [6, 8, 9], we call continuous function sequence Φ additive potential if the following quality holds,

$$\varphi_{n+m}(x)=\varphi_n(x)+\varphi_m(T^n x),\ \text{for all}\ n\geq 1,\ m\geq 1,\ x\in X.$$

It is seen that $\varphi_n(x)=\sum_{i=0}^{n-1}\varphi_1(T^i x)$ for additive potential Φ.

We call Φ almost additive potential if the following conditions holds,

1. $\varphi_n(x)$ is continuous from X to R,
2. There is a positive constant $C(\Phi)$ such that

$|\varphi_{n+p}(x)-\varphi_n(x)-\varphi_p(T^n x)|\le C(\Phi)\ \forall n,p\in \mathbb{N}, \forall x\in X.$

It is obvious that Φ is additive if Φ is almost additive with $C(\Phi)=0$. The almost additive potential arises naturally in the study on the dimension of invariant set in non-conformal repellers [2] and topological pressure of products of positive matrices [7]. Φ is called asymptotically additive potential if for any given $\varepsilon>0$, there exists $g\in C(X,\mathbb{R})$, such that

$$\limsup_{n\to\infty}\frac{1}{n}\|\varphi_n-S_n g_\varepsilon\|<\varepsilon,\ \text{where } \|f\|=\sup_{x\in X}|f(x)| \text{ for } f\in C(X,\mathbb{R}).$$

Lemma A.5 in [9] means that almost additive potential is asymptotically additive potential. However in example 1 in [22], authors give an asymptotically additive potential which is not almost additive. Moreover we can see that asymptotically additive potential arise naturally in the study of dimension theory.

For any given asymptotically additive potential Φ, let

$$\mathrm{H}(\Phi;X,T):=\{x\in X:\lim_{n\to\infty}\frac{1}{n}\varphi_n(x)\text{ does not exist.}\}$$

be the historic set of Φ and

$$\Phi_*(\mu):=\lim_{n\to\infty}\frac{1}{n}\int_X\varphi_n d\mu,\ \text{where } \mu \text{ is a } T\text{-invariant measure.}$$

By asymptotically additivity, the above limit exists. In this paper, we denote the set of T-invariant borel probability measure and ergodic T-invariant borel probability measure by $\mathrm{M}(X,T)$ and $\mathrm{E}(X,T)$ respectively. We write $\Omega_\Phi:=\{\Phi_*(\mu):\mu\in\mathrm{M}(X,T)\}$. Now, we give the result of this paper.

1.4. Main result

In this paper, we prove that the historic set of the asymptotically additive potential in the systems defined in section 1.1 carries full Hausdorff dimension unless it is empty set. By convention, we denote the Hausdorff dimension of A by $\dim_H A$.

Theorem 1 Let $T:\Lambda\to\Lambda$ be $C^{1+\gamma}$ continuous map as in section 1.2 and Π be asymptotically additive potential, then only one of the two cases holds,

1. $\mathrm{H}(\Phi;X,T)=\varnothing$, which is exactly equivalent to Ω_Φ contains only one singleton,
2. $\dim_H \mathrm{H}(\Phi;X,T)=\dim_H\Lambda$.

Remark 1 As point out in Section 1.3, almost additive means asymptotically additive, however, asymptotically additive need to be almost additive. Then our result covers the conclusion in [16].

2. Preliminaries

In this section, we will introduce some lemmas and give the necessary notations.

Recall $A=\{1,2,\cdots,m\}$ is the alphabet, $\Sigma=A^{\mathbb{N}}$ is the full shift and $\Sigma_n=\{w=w_1, w_2,\cdots w_n : w_i\in A\}$ be the words with length n. For $\omega=\{\omega_n\}_{n=1}^\infty\in\Sigma$, write $\omega|_n=\omega_1,\cdots\omega_n$. For $w\in\Sigma_n$, let $[w]:=\{\omega\in\Sigma:\omega|_n=w\}$ be the n-cylinder. For given $\varphi:\Sigma\to\mathbb{R}$ continuous, let $\mathrm{Var}_n\varphi:=\sup_{\omega|_n=\tau|_n}|\varphi_n(\omega)-\varphi_n(\tau)|$ be the n-Variance of Φ and $\|\varphi\|:=\sup_{\tau\in\Sigma}|\varphi(\tau)|$.

Let $\Lambda:=\{x\in\Lambda:\#\{\Pi^{-1}(x)=2\}\}$ be the elements with two coding, where $\# A$ is the cardinality of A. In our settings, Δ and $\Pi^{-1}\Delta$ both are countable set and $\Pi:\Sigma\setminus\Pi^{-1}\Lambda\to\Lambda\setminus\Lambda$ is bijection.

For given word $w=w_1\ldots w_n$, we write $I_n(w)=T_{w_1}\circ\ldots\circ T_{w_n}([0,1])$ and for $\omega\in\Sigma$, also we write $I_n(\omega)=I_n(\omega|_n)$.

Let $D_n(\omega):=\mathrm{diam}(I_n(\omega))$ be the diameter of $I_n(\omega)$, $g(\omega)=-\log T'_{\omega_1}\Pi(\sigma\omega)$ and

$$A_n g(\omega):=\frac{1}{n}\sum_{i=0}^{n-1}g\circ T^i(\omega).$$

By lemma 1 in [11] and lemma 2.1 in [21], $A_n g(\omega)$ can approximate $D_n(\omega)$ uniformly.

Lemma 1. ([11, 21])
Let T be the non-uniformly expanding map defined in section 1.2. Then $D_n(\omega)$ converges to 0 uniformly. Moreover, we have

$$\lim_{n\to\infty}\sup_{\omega\in\Sigma}\left\{|-\frac{1}{n}\log D_n(\omega)-A_n u(\omega)|\right\}=0.$$

Write $\lambda_n(\omega):=-\frac{1}{n}\log D_n(\omega)$. For given σ-invariant measure μ, let $\lambda(\mu,\sigma):=\int_\Sigma g d\mu$ be the lyapunov exponent of μ and $\Pi_*\mu:=\mu\circ\Pi^{-1}$ be the image measure of μ. The following lemma combined by lemma 2 and lemma 3 in [11], plays a key role in the proof of Theorem 1.

Lemma 2. For any given $\mu\in \mathrm{M}(\Sigma,\sigma)$, there exists a sequence of ergodic σ-invariant measures $\{\mu_n\}_{n=1}^{\infty}$ such that $\mu_n\to\mu$ in the weak star topology and $h(\mu_n,\sigma)\to h(\mu,\sigma)$, $\lambda(\mu_n,\sigma)\to\lambda(\mu,\sigma)$.
Next lemma means that the map $\Phi*: M(\Sigma,\sigma)\to \mathrm{R}$ is continuous.

Lemma 3. For any given asymptotically additive potential Φ, let μ be a σ-invariant measure and $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of σ-invariant measures converging to μ in weak star topology, then

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{m}\int\phi_m d\mu_n=\lim_{m\to\infty}\frac{1}{m}\int\phi_m d\mu,$$

i.e. $\lim_{n\to\infty}\Phi_*(\mu_n)=\Phi_*(\mu)$.
By definition of asymptotically additivity, we can check the following result easily.

Lemma 3. Let Φ be asymptotically additive sequence, if $\mathrm{H}(\Phi;X,T)\neq\varnothing$, then we have $\#\Omega_\Phi\geq 2$.

Proof of Theorem 1
Proof. We write $\Psi=\Phi\circ\Pi$, it is easy to check that $\Omega_\Phi=\Omega_\Psi$ and $\mathrm{H}(\Phi;X,T)=\Pi\mathrm{H}(\Psi;X,T)$. Then Theorem 1 is a direct consequence of the following result.

Lemma 5. For any given σ-invariant measures μ and ν such that $\lambda(\mu,\sigma)>0$, $\lambda(\nu,\sigma)>0$ and $\Psi_*(\mu)\neq\Psi_*(\nu)$, we have

$$\dim_H \mathrm{H}(\Phi;X,T)\geq\min\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)},\frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\right\}.$$

Since map T is $C^{1+\gamma}$ continuous, by the lemma 4.6 in [21], we have

$$\dim_H\Lambda=\sup_{\mu\in\mathrm{M}(\Sigma,\sigma)}\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}:\lambda(\mu,\sigma)>0\right\}.$$

For any given $\epsilon>0$, there exists σ-invariant measure μ, such that

$$\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}\geq\dim_H\Lambda-\epsilon.$$

Write $\alpha=\Psi_*(\mu)$, since $\#\Omega_\Phi\geq 2$, there exists σ-invariant measure ν such that $\Psi_*(\nu)=\beta\neq\alpha$. For $0<s<1$, let $\mu_s=s\mu+(1-s)\nu$, then $\Psi_*(\mu_s)\neq\alpha$. By lemma [21], for all $0<s<1$, the following inequality holds,

$$\begin{aligned}\dim_H\mathrm{H}(\Phi;X,T)&\geq\min\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)},\frac{h(\mu_s,\sigma)}{\lambda(\mu_s,\sigma)}\right\}\\&=\min\left\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)},\frac{sh(\mu,\sigma)+(1-s)h(\nu,\sigma)}{s\lambda(\mu,\sigma)+(1-s)\lambda(\nu,\sigma)}\right\}.\end{aligned}$$

Takes s going to 1, by the arbitrary of ϵ, We prove Theorem 1.
Now, we turn to prove lemma 5.

The idea of proof Lemma 5 is to construct a Moran set M whose image ΠM sitting in historic set $\mathrm{H}(\Phi; X, T)$ such that the Hausdorff dimension of ΠM satisfies the following inequality,

$$\dim_H \Pi M \ge \min\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}, \frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\}$$

.

For the convenience of reader, we arrange the proof by 4 steps.

Step 1 Constructing Moran set in odd levels and even levels

By lemma 1, we can take real number sequence $\{\epsilon_i\}_{i=1}^{\infty}$ decreasing to 0 and such that for all $n \ge 2i-1$, the following inequality holds,

$$\mathrm{Var}_n A_n g < \epsilon_{2i-1}, \quad \max_{\omega\in\Sigma} |\tilde{\lambda}_n(\omega) - A_n g(\omega)| < \epsilon_{2i-1}. \tag{1}$$

Similarly for all $n \ge 2i$, we have

$$\mathrm{Var}_n A_n g < \epsilon_{2i}, \quad \max_{\omega\in\Sigma} |\tilde{\lambda}_n(\omega) - A_n g(\omega)| < \epsilon_{2i}. \tag{2}$$

By lemma 2 and lemma 3, we can choose σ -invariant ergodic measure sequence $(\mu_{2i-1})_{i\ge1}$, such that

$$\begin{cases} |\Psi_*(\mu_{2i-1}) - \alpha| < \epsilon_{2i-1}, \\ |h(\mu_{2i-1},\sigma) - h(\mu,\sigma)| < \epsilon_{2i-1}, \\ |\lambda(\mu_{2i-1},\sigma) - \lambda(\mu,\sigma)| < \epsilon_{2i-1}. \end{cases} \tag{3}$$

Similarly, there exists σ -invariant ergodic measure sequence $(\nu_{2i})_{i\ge1}$, such that

$$\begin{cases} |\Psi_*(\nu_{2i}) - \beta| < \epsilon_{2i}, \\ |h(\nu_{2i},\sigma) - h(\nu,\sigma)| < \epsilon_{2i}, \\ |\lambda(\nu_{2i},\sigma) - \lambda(\nu,\sigma)| < \epsilon_{2i}. \end{cases} \tag{4}$$

Write $\alpha_{2i-1} = \Psi_*(\mu_{2i-1})$, noting that the ergodicity of μ_{2i-1}, by Birkhoff ergodic Theorem and Shanon-Mcmillan-Breiman Theorem, for μ_{2i-1} a.e. $\omega \in \Sigma$, we have

$$\begin{cases} \frac{1}{n}\psi_n(\omega) \to \alpha_{2i-1}, \\ A_n g(\omega) \to \lambda(\mu_{2i-1},\sigma), \\ -\frac{\log \mu_{2i-1}[\omega|_n]}{n} \to h(\mu_{2i-1},\sigma). \end{cases} \tag{5}$$

Similarly we have

$$\begin{cases} \frac{1}{n}\psi_n(\omega) \to \alpha_{2i}, \\ A_n g(\omega) \to \lambda(\nu_{2i},\sigma), \\ -\frac{\log \nu_{2i}[\omega|_n]}{n} \to h(\nu_{2i},\sigma) \end{cases} \tag{6}$$

For any given $\delta > 0$, by Ergroff Theorem, there exists $\Omega'(2i-1) \subset \Sigma$ such that $\mu_{2i-1}(\Omega'(2i-1)) > 1-\delta$ and (5) and (6) holds uniformly on $\Omega'(2i-1)$. Then there exists $l_{2i-1} \ge 2i-1$, such that for all $n \ge l_{2i-1}$ and $\omega \in \Omega'(2i-1)$, the following inequality holds,

$$\begin{cases} |\frac{1}{n}\psi_n(\omega)-\alpha_2 i-1|<\epsilon_{2i-1}, \\ |A_n g(\omega)-\lambda(\mu_{2i-1},\sigma)|<\epsilon_{2i-1}, \\ |-\frac{\log\mu_{2i-1}[\omega|_n]}{n}-h(\mu_{2i-1},\sigma)|<\epsilon_{2i-1}. \end{cases} \tag{7}$$

Similarly, there exists $\Omega'(2i)\subset\Sigma$ such that $\nu_{2i-1}(\Omega'(2i))>1-\delta$ and also there exists $l_{2i}\ge 2i$ such that for all $n\ge l_{2i}$ and $\omega\in\Omega'(2i)$, the following inequality holds,

$$\begin{cases} |\frac{1}{n}\psi_n(\omega)-\beta_{2i}|<\epsilon_{2i}, \\ |A_n g(\omega)-\lambda(\nu_{2i},\sigma)|<\epsilon_{2i}, \\ |-\frac{\log\nu_{2i}[\omega|_n]}{n}-h(\nu_{2i},\sigma)|<\epsilon_{2i}. \end{cases} \tag{8}$$

Let $\Sigma(2i-1)=\{\omega|_{l_{2i-1}}\mid\omega\in\Omega'(2i-1)\}$ and

$$\Omega(2i-1)=\bigcup_{\omega\in\Sigma(2i-1)}[\omega].$$

Call $\Omega'(2i-1)$ as $(2i-1)$-level Moran block. Then we have

$$\mu_{2i-1}(\Omega(2i-1))\ge\mu_{2i-1}(\Omega'(2i-1))\ge 1-\delta.$$

Similarly, let $\Sigma(2i)=\{\omega|_{l_{2i}}\mid\omega\in\Omega'(2i)\}$ and

$$\Omega(2i)=\bigcup_{\omega\in\Sigma(2i)}[\omega].$$

We have

$$\nu_{2i}(\Omega(2i))\ge\nu_{2i}(\Omega'(2i))\ge 1-\delta.$$

Inequality (1)-(8) is necessary in following proof.

Step 2 Constructing Moran set contained in historic set

Take $N_0=1$, for $i\ge 1$, let $N_i=2^{l_{i+2}+N_{i-1}}$. We point out that N_i is the repeated times of i-level Moran block $\Sigma(i)$. Let Moran set M is be the following set,

$$\underbrace{\Sigma(1)\cdots\Sigma(1)}_{N_1}\cdots\underbrace{\Sigma(i)\cdots\Sigma(i)}_{N_i}\cdots.$$

For $i\ge 1$, let $\mu_j=\sum_{i=1}^{j}l_iN_i$. For fixed $\omega\in M$, noting that

$$\lim_{j\to\infty}\frac{l_2N_2+l_4N_4+\ldots+l_{2j}N_{2j}}{n_{2j+1}}=0.$$

Using (1), (2), (4), (5), (6), (8), by direct computation, we get the following results,

$$\lim_{j\to\infty}\frac{\psi_{n_{2j+1}}(\omega)}{n_{2j+1}}=\alpha,$$

$$\lim_{j\to\infty}\frac{\psi_{n_{2j}}(\omega)}{n_{2j}}=\beta.$$

These means that $\Pi M\subset\mathrm{H}(\Phi;X,T)$.

Step 3 Constructing Moran measure

For clarity, we relabel Moran block $\Sigma(i)$. Take $\eta_i = \mu_i$ if i is odd and $\eta_i = \nu_i$ if i is even. We rewrite the following integer sequence.

$$\underbrace{l_1,\cdots l_1,}_{N_1}\cdots,\underbrace{l_i,\cdots,l_i,}_{N_i}\cdots.$$

As $(l_i^*)_{i\geq 1}$.

Similarly, we get sequence $\{\Sigma^*(i)\}_{i=1}^{\infty}$, $\{\Omega'*(i)\}_{i=1}^{\infty}$, $\{\Omega*(i)\}_{i=1}^{\infty}$, $\{\eta_i^*\}_{i=1}^{\infty}$, $\{\grave{o}_i^*\}_{i=1}^{\infty}$.

For any given $n \geq 1$, there exists integer $J(n)$, such that

$$\sum_{i=1}^{J(n)} l_i^* \leq n < \sum_{i=1}^{J(n)+1} l_i^*.$$

Also there exists a unique integer $r(n)$, such that

$$\sum_{i=1}^{r(n)} N_i \leq J(n) < \sum_{i=1}^{r(n)+1} N_i.$$

By the definition of $J(n)$, we have the following inequality

$$J(n) \leq J(n+1) \leq J(n)+1, l_{J(n)+1}^* = l_{r(n)+1} \text{ and } l_{J(n)+2}^* \leq l_{r(n)+2}. \quad (9)$$

Noting that $N_i = 2^{l_{i+2}+N_{i-1}}$ for $i \geq 1$, we have

$$l_{J(n)+j}^* / \sum_{i=1}^{J(n)} l_i^* \to 0, \text{ and } \sum_{i=1}^{J(n)+1} l_i^* / \sum_{i=1}^{J(n)} l_i^* \to 1. \quad (10)$$

Now, we define Moran measure. For $\omega \in \Omega^*(i)$, let

$$\rho_w^i = \frac{\eta_i^*[w]}{\eta_i^*(\Omega^*(i))}.$$

It is obvious that $\sum_{w\in\Sigma^*(i)} \rho_w^i = 1$. Write $\mathrm{C}_n := \{[w] : w \in \prod_{i=1}^{n} \Sigma^*(i)\}$, let $\sigma\{\mathrm{C}_n : n \geq 1\}$ be the σ-algebra generated by $\{\mathrm{C}_n : n \geq 1\}$.

For $[w] = [w_1 \cdots w_n] \in \mathrm{C}_n$, define

$$\hat{\eta}([w]) := \prod_{i=1}^{n} \rho_{w_i}^i.$$

Let η be the Kolmogrov extension of $\hat{\eta}$ to all the borel set of M. It is seen that η is a measure on M and η is supported on M.

Step 4 Estimating the lower bound of Hausdorff dimension of Moran measure

In this step, we will show that for all $x \in \Pi M$, the following inequality holds,

$$\liminf_{r\downarrow 0} \frac{\log \Pi_*\eta(B(x,r))}{\log r} \geq \min\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}, \frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\}. \quad (11)$$

By inequality (11), we have

$$\dim_H \Pi_*\eta \geq \min\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)}, \frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\}.$$

Where $\liminf\limits_{r\downarrow 0}\dfrac{\log \Pi_*\eta(B(x,r))}{\log r}$ is the lower local dimension of $\Pi_*\eta$ at x and

$$\dim_H \Pi_*\eta := \sup\{s\ge 0:\ \liminf_{r\downarrow 0}\frac{\log \Pi_*\eta(B(x,r))}{\log r}\ge s \text{ for } \Pi_*\eta \text{ a.e. } x\in\Lambda\}$$

is the Hausdorff dimension of measure $\Pi_*\eta$. As for this part, the reader can refer to the Chapter 10 in [5]. Noting that $\mathrm{H}(\Phi;X,T)\supset \Pi M$ and η supported on M , we have

$$\dim_H \mathrm{H}(\Phi;X,T)\ge \dim_H \Pi M\ge \dim_H \Pi_*\eta \ge \min\{\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)},\frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)}\}.$$

Thus, Lemma 5 follows. Now, we go to prove inequality (11).

Firstly, we estimate the lower bound of $D_n(\omega)$ for $\omega\in M$. Let $\tau_i=\mu$ for i is odd and $\tau_i=\nu$ for i is even. By (1), (4), (2), (5), (6), (8), we have

$$n\tilde{\lambda}_n(\omega)\le \sum_{i=1}^{J(n)} l_i^*(\lambda(\tau_i,\sigma)+4\epsilon_i^*)+l_{J(n)+1}^*(\|g\|+\epsilon_{J(n)}^*):=\rho(n).$$

It is obvious that $\rho(n)$ is increasing and $D_n(\omega)\ge e^{-\rho(n)}$.

For fixed $x\in \Pi M$ and small $r>0$, there exists unique integer $n=n(r)$ such that $e^{-\rho(n+1)}\le r<e^{-\rho(n)}$. We define the following set,

$$\mathrm{C}:=\{I_n(\omega):\omega\in M \text{ and } \mathrm{I_n}(\omega)\cap B(x,r)\ne\varnothing\}.$$

Since $D_n(\omega)\ge \mathrm{e}^{-\rho(n)}$, then there exists at most 3 elements in C. For any given $\omega\in M$ with $I_n(\omega)\in \mathrm{C}$, we have $\omega|_n = w_1\cdots w_{J(n)}v$, where v is the prefix of some element in $\Sigma^*(J(n)+1)$.

Then we have

$$\begin{aligned}\log \Pi_*\eta(B(x,r)) &\le -\sum_{i=1}^{J(n)} l_i^*\left(-\frac{\log \eta_i^*[w_i]}{l_i^*}\right)-(J(n)+1)\log(1-\delta)+\log 3\\ &\le -\sum_{i=1}^{J(n)} l_i^*(h(\tau_i,\sigma)-2\epsilon_i^*)-(J(n)+1)\log(1-\delta)+\log 3.\end{aligned}$$

By the inequality above, meanwhile noting that $r\to 0$ iff $n\to\infty$, we have

$$\begin{aligned}\liminf_{r\downarrow 0}\frac{\log \Pi_*\eta(B(x,r))}{\log r} &\ge \liminf_{n\to\infty}\frac{\sum_{i=1}^{J(n)} l_i^*(h(\tau_i,\sigma)-2\epsilon_i^*)+(J(n)+1)\log(1-\delta)-\log 3}{\sum_{i=1}^{J(n+1)} l_i^*(\lambda(\tau_i,\sigma)+4\epsilon_i^*)+l_{J(n+1)+1}^*(\|g\|+\epsilon_{J(n+1)+1}^*)}\\ &=\liminf_{n\to\infty}\frac{\sum_{i=1}^{J(n)} l_i^*(h(\tau_i,\sigma)-2\epsilon_i^*)}{\sum_{i=1}^{J(n)} l_i^*(\lambda(\tau_i,\sigma)+4\epsilon_i^*)}\\ &\ge \liminf_{n\to\infty}\frac{\sum_{i=1}^{J(n)} l_i^*(\lambda(\tau_i,\sigma)+4\epsilon_i^*)\min\left\{\frac{h(\mu,\sigma)-2\epsilon_i^*}{\lambda(\mu,\sigma)+4\epsilon_i^*},\frac{h(\nu,\sigma)-2\epsilon_i^*}{\lambda(\nu,\sigma)+4\epsilon_i^*}\right\}}{\sum_{i=1}^{J(n)} l_i^*(\lambda(\tau_i,\sigma)+4\epsilon_i^*)}\\ &\ge \min\frac{h(\mu,\sigma)}{\lambda(\mu,\sigma)},\frac{h(\nu,\sigma)}{\lambda(\nu,\sigma)},\end{aligned}$$

where the last inequality follows from the following fact.

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two positive integer sequence, then we have

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} \geq \liminf_{n\to\infty} \frac{a_n}{b_n}.$$

Thus, we get inequality (11).

3. References

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