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A new proof of the chain rule for derivatives of compound functions

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Abstract
This article gives a new proof of the chain rule for derivatives of compound functions which is stricter than the former proof in text book.

Keywords: The chain rule, derivatives of compound functions, new proof

1. Introduction
The chain rule for derivatives of multivariate functions is a fundamental theorem in basic mathematics, and is also a significant and useful means of finding derivative of the functions of both one variable and several variables. The text book had given its proof, but it is not completely rigorous [1, 2, 3]. Therefore, this article will give a respectively strict proof.

2. The proof of the chain rule for derivatives of univariate compound function
Theorem 1. if the function $u=g(x)$ is differentiable at point x_0, and function $y=f(u)$ is also differentiable at point $u_0=g(x_0)$, then the compound function $y=f(g(x))$ is differentiable at x_0, and its derivative is $\frac{dy}{dx}=f'(u_0)\cdot g'(x_0)$ or $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

Proof. $\because y=f(u)$ is differentiable at u_0

$$\therefore \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=\lim_{\Delta u\to 0}\frac{f(u_0+\Delta u)-f(u_0)}{\Delta u}=f'(u_0)$$

$$\therefore \frac{\Delta y}{\Delta u}=f'(u_0)+\alpha$$

when $\Delta u\to 0$, $\alpha\to 0$

$$\therefore \Delta y=f'(u_0)\Delta u+\alpha\Delta u$$

Similarly $\Delta u=g'(x_0)\Delta x+\beta\Delta x$

When $\Delta x\to 0$, $\beta\to 0$

$$\therefore \Delta y=[f'(u_0)+\alpha][g'(x_0)+\beta]\Delta x$$

$$\therefore \frac{\Delta y}{\Delta x} = [f'(u_0)+\alpha][g'(x_0)+\beta] = f'(u_0)g'(x_0)+\beta\cdot f'(u_0)+\alpha\cdot g'(x_0)+\alpha\beta$$

The following will prove that when $\Delta x \to 0$, $\Delta u \to 0$,so $\alpha \to 0$ and $\beta \to 0$

Assuming that when $\Delta x \to 0$, Δu does not tend to 0.

Then $\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \infty$

So $u(x)$ is not differentiable at x_0 which conflicts with the condition. Then the assumption is incorrect.

S

o when $\Delta x \to 0$, $\Delta u \to 0$, so $\alpha \to 0$ and $\beta \to 0$

$$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0)$$

3. The proof of the chain rule for derivatives of multivariate compound function

Theorem 2. if function $u=\varphi(x,y)$, $v=\psi(x,y)$ are differentiable at point $(x_0,y_0)\in D$ (D is the definition flied of both function $u=\varphi(x,y)$ and $v=\psi(x,y)$), and function $z=f(u,v)$ is differentiable at point $(u_0,v_0)=(\varphi(x_0,y_0),\psi(x_0,y_0))$, then the compound function $z=f(\varphi(x,y),\psi(x,y))$ is differentiable at point (x_0,y_0), and its Partial derivatives of x and y are:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Proof. $\because$ The outer function $z=f(u,v)$ is differentiable at point

$$(u_0,v_0)=(\varphi(x_0,y_0),\psi(x_0,y_0))$$

$\therefore$ Its total differential at this point is $\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + o(\rho)$

$$\rho = \sqrt{(\Delta u)^2+(\Delta v)^2}$$

Let $o(\rho)=\alpha\rho$, $\lim_{\substack{\Delta u\to 0\\ \Delta v\to 0}}\alpha = 0$

$$\therefore \frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\frac{\Delta u}{\Delta x}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta x}+\frac{\alpha\rho}{\Delta x}$$

$\because u=\varphi(x,y)$, $v=\psi(x,y)$ are differentiable at (x_0,y_0)

$\therefore$ When $\Delta x \to 0$

$$\frac{\Delta u}{\Delta x}\to\frac{\partial u}{\partial x},\quad \frac{\Delta u}{\Delta y}\to\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial z}{\partial x} = \lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\lim_{\Delta x\to 0}\frac{\alpha\rho}{\Delta x}$$

$$\because \lim_{\Delta x\to 0}\frac{\alpha\rho}{\Delta x} = \lim_{\Delta x\to 0}\alpha\sqrt{(\frac{\Delta u}{\Delta x})^2+(\frac{\Delta v}{\Delta x})^2}$$

And $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \neq \infty$, $\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{du}{dx} \neq \infty$

$\therefore \sqrt{(\frac{\Delta u}{\Delta x})^2 + (\frac{\Delta v}{\Delta x})^2}$ is a bounded quantity.

$\because \lim_{\Delta x \to 0} \alpha = 0$ $\therefore \lim_{\Delta x \to 0} \frac{\alpha \rho}{\Delta x} = 0$

$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$, similarly $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$

4. Conclusion

According to the proofs above, it can be concluded that the proofs of the chain rule for derivatives of compound functions are diverse. Additionally, these two proofs avoid some vague places which need further explanation. For instance, the original proof have to make a complementary definition that when $\Delta u = 0$, $\alpha = 0$ [4]. So comparatively the proofs of the chain rule for derivatives of compound functions given above is simpler and stricter.

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