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Walter Onchere
Department of Mathematics and Actuarial Science, Kisii University, Kisii, Kenya

Calvin Maina
Department of Mathematics and Actuarial Science, Kisii University, Kisii, Kenya

Lameck Agasa Ondieki
Department of Mathematics and Actuarial Science, Kisii University, Kisii, Kenya

Correspondence
Walter Onchere
Department of Mathematics and Actuarial Science, Kisii University, Kisii, Kenya

Frailty mixture model with application in insurance industry

Walter Onchere, Calvin Maina and Lameck Agasa Ondieki

Abstract
Heterogeneity in a population of assured lives in respect of mortality can be explained by differences among the individuals; some of these are observable, while others, for instance genetic factors having influence on survival are difficult to measure. This undermines usage of observable risk factors as the only rating factors for life insurance. This heterogeneity exposes insurers to adverse selection if only the healthiest lives purchase annuities, so standard annuities are priced with a mortality table that assumes above-average longevity. This makes standard annuities expensive for many individuals. To avoid biases in valuation a better understanding of heterogeneity in required.
Frailty models are extensions of the Cox proportional hazards model which is popular in survival studies. The frailty approach is a statistical modeling method which aims to account for the heterogeneity caused by unmeasured covariates. It does so by adding random effects which act multiplicatively on the hazard. In this paper we consider the gamma, inverse gaussian and non-central gamma as frailty distributions with Weibull distribution as the baseline. The results shows that, the non-central gamma frailty model is appropriate for representation of the insurer's liability when heterogeneity is present.

Keywords: Gamma distribution, non-central gamma distribution, inverse Gaussian distribution, weibull distribution, Markov chain Monte Carlo, pension scheme

Introduction
The concept of frailty modeling is based on mixture distributions and survival analysis.

Varying parameter and unknown covariates
Johnson *et al*. (2005, p.345) [11], describes a mixture distribution as a distribution that arises when the density function of a random variable depends on a parameter. Consider a random variable x depending on its parameter θ then the conditional probability density function can be written as:

$$f(x) = \int f(x|\theta)g(\theta)\, d\theta$$

Thus instead of considering a parameter θ , we consider an unknown covariate.

Survival analysis
Survival analysis is a branch of statistics which deals with time to the occurrence of a given event of interest. For insurers this event could be time to death, ill health or retirement. It's different from other fields of statistics in that we are observing something that develops dynamically over time and takes censoring into consideration which is partial information about the variable of interest.
Three important functions of time to the event are:

- The survival function, $S(t)$, describes the probability that an individual survives longer than time t.
- The probability density function, $f(t)$
- The hazard function, $h(t)$, describes the instantaneous death rate

Estimating the survival function using non-parametric methods such as the Kaplan-Meier technique leads to obtaining the median time to the event under investigation.
Determining factors affecting the hazard, the Cox PH model has been widely used.

Estimating the survival function
Existing literature on estimating the survival function includes the parametric, semi-parametric and non-parametric methods. Frailty models are extension of Cox-proportional hazard model where the relative risk is replaced with a random variable called the 'frailty term'.

Parametric Methods
For parametric inference, it is necessary to make assumptions about the distribution of failure times. Parametric approaches such as Weibull, lognormal, exponential, etc can be used to estimate the survival function for homogeneous populations. Basically, any distribution of non-negative random variables can be used.

Non-Parametric Methods
Non-parametric approaches such as Aalen-Nelson (1978) [2] can be used to estimate the survival function when assumption of the failure time distribution is to be avoided. An advantage of non-parametric models is their good fit and their ability to deal with any distribution without any additional assumptions.

Semi-Parametric Methods
Cox proportional hazard model (1972)
A Cox model is a technique for exploring the relationship between the survival time of an individual and several explanatory variables. The hazard function for each individual is proportional to the baseline hazard $h_o(t)$ and thus the hazard is fully determined by the covariate vector. The hazard function for individual i at time (age) t is written as:

$$h_i(t) = h_o(t)\, e^{\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k}$$

$h_o(t)$ is the baseline hazard function and corresponds to the probability of dying (or reaching an event) when all the explanatory variables are zero. In this model it is left unspecified. $exp(\beta' X_i)$ is the relative risk of individual i, where $x_i = (x_1, \ldots, x_k)$ are vector of covariates and $\beta' = (\beta_1, \ldots, \beta_k)$ are vector of regression coefficients that give the proportional change that can be expected in the hazard, related to changes in the explanatory variables.

Frailty Models
Vaupel *et al*. (1979) [17] introduced the term frailty and used it in individual survival models. Clayton (1978) [6] promoted the model by its application to multivariate situations. Ordinary life table methods implicitly assume that the population under study is homogenous. This means that all individuals in that study are subject under the same risk (e.g., risk of death, risk of accident). Basic observation of medical statistics shows that individuals differ greatly. Thus, the study population cannot be assumed to be homogeneous but must be considered as a heterogeneous sample.
A random effect model takes into account the effects of unobserved or unobservable heterogeneity, i.e. an individual's attitude towards health or some genotypic personal characteristics. Thus, the role of "frailty" is to include all unobservable factors acting on the individual mortality. The random effect denoted by Z is the term that describes the individual heterogeneity.

Differential Mortality Models
The Multiplicative approach.
Vaupel *et al*. (1979) [17] described the model as

$$h(x|z) = z h_o(x)$$

$h_o(x)$ is the baseline mortality considered to be a known function of x that is to be specified. The frailty, Z is meant to quantify uncertainty associated with the hazard rate which acts in a multiplicative manner

The Aalen Additive Model.
Aalen (1978) [2] described a nonparametric additive hazard model given by

$$h(x|Z) = h_o(x) + \beta Z \quad \beta > 0$$

This model is useful in dealing with right censored survival data, especially in the presence of time-varying covariates.

Age Shifting Model

$$h'(x) = h(x + z)$$

The age shift model was proposed by G Humphreys (1874). Who argued that the mortality experience of a group of impaired lives accepted for life insurance should have an increased premium rating determined by assuming that the insured's age is higher than the real current age, hence adopting the "age shift".

Types of Frailty Models

Frailty models without observed covariates.

This model is used when only survival data is available for the analysis, or when additional information is of no interest.

$$h(t,Z) = Zh_o(t)$$

The non-negative random variable Z is called frailty and $h_o(t)$ is the baseline hazard. This model is non-identifiable from survival data, since different combinations of $h_o(t)$ and frailty distributions may produce the same marginal hazard rate $h(t)$. The model becomes identifiable when the parametric structure of $h_o(t)$ is fixed and Z is assumed to belong to some parametric distribution family.

Univariate Frailty Models

This model describes the influence of unobserved covariates in a proportional hazards model for independent lifetimes. The variability can be split into a part that depends on observable risk factors, and is therefore theoretically predictable, and a part that is theoretically unpredictable, even when all relevant information is known. This model has been used by Hougaard (1984) [10] to show that these two sources of variability can explain some unexpected results or gives an alternative explanation of some results.

Literature Review

Frailty Distributions

First, neither theory nor data typically provides much guidance for choosing a specific distribution from which to draw the frailty, thus any distribution with positive support and finite mean is suitable to represent the frailty distribution. However, for tractability reasons the choice of distribution is limited to those that provide a closed form expression for the frailty survivor function, density and hazard functions.

The choice of parametric distributions for Z is often a matter of computational convenience and it should be strictly positive support, since negative frailty leads to negative mortality rates. Some of the distributions considered in this study are:

- Gamma distribution Vaupel *et al*. (1979) [17]
- Inverse-Gaussian distribution Manton *et al*. (1986) [12]
- Non Central Gamma distribution

Baseline Hazard Distributions

Two different approaches are possible. In the parametric case the baseline hazard is chosen in the class of parametric lifetime distributions. The model also works without any specification of the baseline hazard function. However, there has been no study or survival experiment, which restricts estimates for the parametric form of the baseline hazard.

The baseline hazard considered in this study is the Weibull distribution (Manton and Stellard, 1986) [12]

Applications in Life Insurance

Frailty models are used in life insurance to represent heterogeneity in a population due to unobservable risk factors. Heterogeneity due to observable risk factors is addressed at policy issue during the underwriting process to ensure that each contract is assigned premium consistent with the insured risk. Neglecting such factors may lead to biased valuation of insurance products.

Actuaries have developed models for valuing life insurance that only consider observable risk factors. However, in general insurance models accounting for unobservable risk has been developed to explain overall claim frequency i.e. the Negative Binomial~Pareto Model [18].

Life Insurance

Life insurance contracts with benefits contingent on the lifetime of an individual and whose benefit is stated in advance is considered. Heterogeneity can be classified as emerging from observable risk factors (at issue) i.e. age, sex, health status, profession, smoke habits, sport activities, and so on. Or unobservable risk factors like an individual’s attitude towards risk.

For immediate annuities, the relation of premium and annuity depends on the health of the insured at the time the contract is taken out. However, in deferred annuities (pension schemes), the insurer has to perform some kind of underwriting at the end of the deferment period.

The underwriting process

The purpose of underwriting is to assign each insured a frailty factor $\tilde{Z}$ as an estimate of Z to determine the pricing mortality rates. These underwriting factors are observable characteristics, such as smoking status, that explain mortality heterogeneity. Underwriting is done to ensure that premiums and benefits are fairly priced.

The tests carried out during the underwriting process are based on.

- Biological and physiological factors, such as age, gender, genotype.
- Features of the living environment; in particular: climate and pollution, nutrition standards population density, hygienic and sanitary conditions.
- Occupation, in particular in relation to professional disabilities or exposure to injury, and educational attainment.
- Individual lifestyle, in particular with regard to nutrition, alcohol and drug consumption, smoke, physical activities and pastimes.
- Current health conditions, personal and/or family medical history, civil status, and so on.

This assessment can be performed through proper questions in the application form and as to health conditions through a medical examination.

Modeling unobservable factors

In addition to observable factors, heterogeneity may be caused by unobservable individual-specific factors, referred to as frailty. Frailty models may provide an appropriate description of the age-specific mortality shape, as well as the estimate of parameters of the relevant models according to mortality observed within the portfolio. However, there is no data available that can be linked to the choice of the distribution of Z since it is unobserved

The rest of this paper is organized as follows; Section 3 describes the frailty model construction and parameter estimation, Section 4 describes an application of frailty modeling in actuarial science and Section 5 is on discussions and recommendation for further research. Finally, Section 6 gives references used in the study

Methodology

Probability Tools

Some of the common probability tools used in survival analysis, Bowers *et al.* (1997) [3] are described below. Let T_x be the future lifetime variable i.e. the remaining duration of life of a person aged x, which is a positive real valued variable, having a continuous distribution with finite expectation. Several functions characterize the distribution of T_x:

- $f_x(t), t \geq 0$ is the probability density of T_x;
- $S_x(t) = P(T_x > t) = \int_t^\infty f_x(x)dx = 1 - F_x(t)$ is the survival function, which is sometimes denoted with ${}_tP_x$
- $F_x(t) = P(T_x \leq t)$ Expresses the probability of dying within t years for a person age x and is denoted by ${}_tq_x$
- $h(t) = \frac{f(t)}{S(t)} = \lim_{\delta t \to 0} \frac{P\ t \leq T < t+\delta t | T \geq t)}{\delta t} = \frac{-\partial S(t)/\partial t}{S(t)}$ is the hazard function, which represents the probability that an individual alive at t experiences the event in the next period σt. (also called the instantaneous death rate)
- $H(t) = \int_0^t h(x)dx$ is the cumulative hazard function

Relationships between $\boldsymbol{f(t), h(t)}$ *and* $\boldsymbol{s(t)}$

Let $h(t)$ denote the hazard function, defined by

$$h(t) = \lim_{(dt \to 0)} \frac{pr(t<T\leq t+dt \mid T>t)}{dt}$$

T is nonnegative and represents the future lifetime of an individual

$$h(t) = \lim_{(dt \to 0)} \frac{pr(t<T\leq t+dt \mid T>t)}{prob(T>t)*dt}$$

$$h(t) = \lim_{(dt \to 0)} \frac{pr(t<T\leq t+dt)/dt}{prob(T>t)}$$

$$h(t) = \frac{f(t)}{1-F(t)}$$

$$h(t) = \frac{f(t)}{S(t)} \qquad \ldots (1)$$

By definition;

$$S(t) = 1 - F(t)$$

$$f(t) = F'(t) = -S'(t)$$

Substituting in (1)

$$h(t) = \frac{-S'(t)}{S(t)}$$

$$h(t) = -\frac{d}{dt} ln(S(t))$$

$$-\int h(t)\,dt = ln\,s(t)$$

$$S(t) = exp(-\int h(t)\,dt)$$

$$S(t) = \exp(-H(t))$$

Laplace Transform

The Laplace transform is crucial in this study since it makes computations of the survival and hazard functions from the density function easy.

The Laplace transform of a random variable Z with density function $f(z)$ is given by;

$$L_Z(s) = E[e^{-sZ}]$$

$$L_Z(s) = \int e^{-sZ} f(z)dz$$

Frailty Models

The frailty approach aims to account for heterogeneity, caused by unmeasured covariates in the Cox-proportional model which is described by a mixture variable Z called frailty.

The Cox-proportional model is given by;

$$h(t,x) = h_o(t)\, exp(\beta' X).$$

The hazard is modified to the frailty model by substituting the relative risk $exp(\beta' X_i)$ by a random variable Z which represents the unobserved covariates X_i i.e.

$$h(t,z) = h_o(t) * Z$$

The frailty Z is then assumed to follow some distribution with positive support and has a multiplicative effect on the baseline hazard function which is common to all individuals.

The Multiplicative Model

This model describes the population as a mixture and assumes that each individual correspond a frailty quantity Z, describing the individual's relative risk. The non-negative quantity z encompasses all other factors affecting mortality other than age.

The hazard at age x conditional on Z is assumed to be $Z\, h_o(x)$

I.e. $h(x|z) = z * h_o(x)$ where $h_o(x)$ is the 'standard hazard function' corresponding to a 'standard individual', conventionally those with frailty $z = 1$

Individuals with $Z > 1$ experience a force of mortality that is proportionally higher than $h(x)$ at all ages. Individuals with $Z < 1$ experience proportionally lower mortality rates. $Z = 1$ Correspond to the standard hazard function.

The composition of a cohort with respect to the frailty Z changes as a cohort grows older because the more frail (susceptible) individuals tend to die earlier than the least frail individuals.

Due to the stochastic nature of Z, the random effect or frailty model is stochastic.

The survival function of an individual with frailty Z is given by

$$S(t|Z) = exp(-\int h(t|Z)\, dt)$$

$$= exp(-\int Zh_o(t)\, dt)$$

$$S(t|Z) = exp\{-ZH_o(t)\}$$

Since, the individual model $S(t|Z)$ is not observable as each individual Z is unobserved; it is 'integrated out' by specifying a distribution and obtaining the unconditional survival function.

The survival function of the total population is the mean of individual survival functions with respect to the frailty distribution. It can be viewed as the survival function of a randomly drawn individual, and corresponds to what can actually be observed.

Integrating over the range of frailty variable Z having density $f(z)$, we get marginal survival function representing the population as,

$$S(t) = \int S(t|Z)\, f(Z)\, dz$$

$$S(t) = E[S(t|Z)]$$

$$S(t) = E[exp\{-ZH_o(t)\}]$$

$$S(t) = L_Z(H_o(t))$$

$f(z)$ is the density of Z and $L_Z(s)$ is the Laplace transform of Z.

To obtain the marginal density function $f(t)$

Consider the relationship;

$$h(t|Z) = \frac{f(t|Z)}{s(t|Z)} = Zh_o(t)$$

$f(t|Z) = Zh_o(t)S(t|Z)$

Since, $f(t|Z) = \frac{f(t,Z)}{f(z)}$

$f(t,Z) = Zh_o(t)S(t|Z)f(z)$

Also, $f(t) = \int f(t,Z)f(z)\,dz$

$f(t) = h_o(t)\int ZS(t|Z)f(z)dz$ … (2)

$= h_o(t)E[ZS(t|Z)]$

$f(t) = -h_o(t)L'(H_o(t))$

Model Assumptions

- The frailty Z has a multiplicative effect on the mortality rate of the individuals:
- $h(t;Z) = Zh_o(t)$
- The frailty Z_x is stationary. i.e. the frailty of an individual keeps constant throughout the whole lifetime span (but the probability distribution does depend on the age, and this justifies the suffix x)
- Z is distributed independent of age(x) or time(t)
- Z has a strictly positive support since negative hazards are impossible.

Gamma Frailty Model

Vaupel *et al.* (1979) [17] suggest a Gamma distribution, due to its mathematical tractability. From a computational and analytical point of view, it fits well to failure data because it is easy to derive the closed form expressions of unconditional survival, cumulative density and hazard function. This is due to the simplicity of the Laplace transform.

Vaupel Approach (1979) [17]

Construction

$Let\ Z \sim \Gamma(p,b)$

With shape parameter p and scale parameter b. The marginal density of Z is;

$f(z) = \frac{b^p z^{p-1} e^{-bz}}{\Gamma(p)}\ ; z > 0, b > 0, p > 0$

The Laplace transformation is given by;

$L_Z(s) = (\frac{b}{b+s})^p = (1 + \frac{s}{b})^{-p}$

This is required to integrate out the distribution of the unobserved frailty. Once the frailty is integrated out, accounting for unobserved heterogeneity is reduced to estimating the variance of the frailty term. $\delta^2 = \frac{1}{b}$

The mean frailty at birth is

$E(z) = -L'(0) = p * (1 + \frac{s}{b})^{-p-1} * \frac{1}{b}\ @s = 0$

$= \frac{p}{b}$

$Variance;\ Var(z) = L''(0) - (L'(0))2$

$= -p(-p-1) * (1 + \frac{s}{b})^{-p-2} * \left(\frac{1}{b}\right)^2 - \left(\frac{p}{b}\right)^2\ @s = 0$

$= \frac{p}{b^2}$

$Coefficient\ of\ variation;\ cv(z) = \frac{sd}{mean} = \frac{1}{\sqrt{p}}$

The CV shows that p plays the role of measuring, in relative terms, the level of heterogeneity in population. If $p \to \infty$, then $cv(z) \to 0$, i.e. the population can be considered homogeneous; for small values of p, on the contrary, the value of $cv(z)$ is high, expressing a wide dispersion, i.e. heterogeneity in the population.

However, the coefficient of variation is constant and does not change with age. This is a unique property of the gamma distributed frailty, since other assumed forms of frailty usually exhibit a decreasing coefficient of variation. i.e. Inverse Gaussian distributed frailty.

The marginal survival function is given by;

$$S(x) = L_Z(H_0(x))$$

$$S(x) = (1 + \frac{H_0(x)}{b})^{-p}$$

$$f(x) = -h_o(t)\, L_Z{}'(H_0(x))$$

$$f(x) = h_o(t)\,(1 + \frac{H_0(x)}{b})^{-p-1} \frac{p}{b}$$

$$h(x) = \frac{f(x)}{s(x)} = h_o(t)\,(1 + \frac{H_0(x)}{b})^{-1} \frac{p}{b}$$

For purposes of identifiability assume the distribution of Z has mean normalized to one (i.e. the standard mortality table describes an "average individual") and variance$\delta^2 = \frac{1}{b}$.

Let $p = b$ (i.e. one parametric gamma distribution). The hazard becomes,

$$h(x) = \frac{h_o(x)}{1+\frac{H_o(x)}{b}}$$

$$h(x) = \frac{h_o(x)b}{b+H_o(x)}$$

Choice of $h_o(x)$
Weibull-Gamma Frailty Model
$h_o(x)$ is chosen to follow a weibull (λ, p) distribution with probability density function $f(x) = \lambda p x^{p-1} exp(-\lambda x^p)\ where\ p > 0\ , \lambda > 0$ p is the shape parameter

The survival function is;

$$s(x) = \Pr(X > x)$$

$$s(x) = \int_x^{\infty} \lambda p t^{p-1} exp(-\lambda t^p)\, dt \text{ ….. eqt1}$$

$$let\ z = \lambda t^p\ \frac{dz}{dt} = \lambda p t^{p-1}\ substituting\ in\ eqn\ 1$$

$$s(x) = \int_{\lambda x^p}^{\infty} \exp(-z)\, dz$$

$$s(x) = exp(-\lambda x^p)$$

$$h(x) = \frac{f(x)}{s(x)} = \frac{\lambda p x^{p-1} exp(-\lambda x^p)}{exp(-\lambda x^p)}$$

$$h_o(x) = \lambda p x^{p-1}$$

$$H_o(x) = \int_0^x h_o(t)\, dt$$

$$H_o(x) = \int_0^x \lambda p t^{p-1}\, dt$$

$$H_o(x) = \lambda x^p$$

$if\ p > 1$ the hazard increases and if $p < 1$ the hazard decreases

The extreme value character of the Weibull distribution makes it appropriate for the distribution of individual time to death, because there are different causes of death which compete with each other.

Using Weibull as the baseline hazard, the hazard function for gamma;

$$h(x) = \frac{h_o(x)b}{b + H_o(x)}$$

becomes

$$h(x) = \frac{\lambda p x^{p-1} * b}{b + \lambda x^p}$$

Inverse-Gaussian Frailty Model
Alternative to the Gamma distribution is the Inverse Gaussian as a frailty distribution introduced by Hougaard (1984) [10]. When the inverse Gaussian is used, the variability of *Zx* decreases with age which can be justified by the fact that those with low frailty keep on living.

Hougaard Approach (1984) [10]
Construction

$Let\ Z \sim IG(\mu, \Theta)$

The probability density function of Z is

$$f(z, \mu, \Theta) = \left(\frac{\Theta}{2\pi z^3}\right)^{1/2} exp\left\{-\frac{\Theta(z-\mu)^2}{2z\mu^2}\right\} \ for\ z > 0\ \Theta > 0\ \mu > 0$$

Substituting $\Theta = \frac{\mu^2}{\beta}$

$$f(z, \mu, \lambda) = \mu\left(\frac{1}{2\pi\beta z^3}\right)^{1/2} exp\left\{-\frac{(z-\mu)^2}{2z\beta}\right\} \text{ for } z > 0\ \beta > 0\ \mu > 0$$

The Laplace transform is given by;

$$L_Z(s) = \exp\{-\frac{\mu}{\beta}[(1 + 2\beta s)^{\frac{1}{2}} - 1]\}$$

Mean = $-L'_z(0) = \mu$

Variance = $L''_z(0) - \mu^2$
$= \mu\beta$

Coefficient of Variation = $\frac{\sqrt{\beta}}{\mu}$

For identifiability reasons the mean is normalized to one. i.e. $\mu = 1$ Thus the variance $\delta^2 = \beta$

The Laplace transform becomes

$$L_z(s)=\exp\left[\frac{1-(1+2s\delta^2)^{1/2}}{\delta^2}\right]$$

The marginal survival function is given by;

$$S(x) = L_Z(H_0(x))$$

$$= \exp\left[\frac{1-(1+2H_0(x)\delta^2)^{1/2}}{\delta^2}\right]$$

$$f(x) = -h_o(x)\, L_Z{}'(H_0(x))$$

$$= \frac{h_o(x)}{(1+2H_0(x)\delta^2)^{1/2}} \exp\left[\frac{1-(1+2H_0(x)\delta^2)^{1/2}}{\delta^2}\right]$$

$$h(x) = \frac{f(x)}{s(x)} = \frac{h_o(x)}{(1+2H_0(x)\delta^2)^{1/2}}$$

Choice of $h_o(x)$
Weibull-Inverse Gaussian Frailty Model
Using Weibull distribution for $h_o(t)$

$$h_o(t) = \lambda p x^{p-1}$$

$H_0(x) = \lambda x^p$

$$h(x) = \frac{h_o(x)}{(1+2H_0(x)\delta^2)^{1/2}}$$

becomes,

$$h(x) = \frac{\lambda p x^{p-1}}{(1+2\,\lambda x^p \delta^2)^{1/2}}$$

Non-Central Gamma Frailty Model

The probability density function for the non-central gamma distribution with Y being a mixing of the distributions of $X_1, X_2, \dots, X_N$

Where $Xi's \sim Gamma(n,1)\ and\ N \sim poisson(\lambda)$

Then the density function is a convolution with respective weights $\frac{e^{-\lambda}(\lambda)^i}{i!}$ i.e

$Y = X_1, X_2, \dots, X_N$

$$Prob(Y = j) \ = \sum_{j=0}^{\infty} prob\left(X_1, X_2, \dots, X_j \middle| N = j\right) prob(N = j)$$

$$Prob(Y = j) \ = \sum_{j=0}^{\infty} \{\frac{X^{j-1}e^{-X}}{\Gamma(j)}\}^{*n} * \{\frac{\lambda^j e^{-\lambda}}{j!}\}$$

$$Prob(Y = j) \ = \sum_{j=0}^{\infty} \{\frac{X^{n+j-1}e^{-X}}{\Gamma(n+j)}\} * \{\frac{\lambda^j e^{-\lambda}}{j!}\}$$

$$f(x,n,\lambda) = \sum_{j=0}^{\infty} \{\frac{X^{n+j-1}e^{-X}}{\Gamma(n+j)}\} * \{\frac{\lambda^j e^{-\lambda}}{j!}\}$$

Where $\Gamma(n)$ is the central complete gamma function with $n > 0\ \lambda > 0\ x \geq 0$

The hazard function is a special case of the three parameter power variance function when $r = -1$

$$h(t) \ = \frac{h_o(t)}{(1+\frac{1}{2}*\delta^2 H_o(t))^2}$$

Choice of $h_o(x)$

Weibull –Non central Gamma Frailty Model

Using Weibull distribution for $h_o(t)$

$$h_o(t) \ = \ \lambda p x^{p-1}$$

$H_0(x) = \lambda x^p$

The hazard function

$$h(t) = \frac{h_o(t)}{\left(1+\frac{1}{2}*\delta^2 H_o(t)\right)^2}$$

Becomes,

$$h(t) \ = \frac{\lambda p x^{p-1}}{(1+\frac{1}{2}*\delta^2 \lambda x^p)^2}$$

Parameter Estimation

Model parameters are fixed quantitative values that characterize the model believed to reflect the real world. They have to be estimated either by statistical inference from observations or by expert opinion.

Choice of Explanatory Variables

In order to make comparisons between the Gamma-Weibull model, the Inverse-Gaussian-Weibull model and the Non-central Gamma-Weibull model, it is necessary to estimate and fix the baseline model parameters using insurance based mortality rating. The baseline model has no underwriting. The weibull parameters are estimated using MCMC technique see Gamerman, D. (1997) [19] the R-CODE used in this study is shown below (appendix).

Choice of the Insureds Level of Heterogeneity

In Butt and Haberman (2004) [5] an insurance application of frailty-based survival model is proposed. In particular, the authors discuss various choices and fit some models to two sets of life insurance mortality data. The obtained results suggest that when life annuities are referred to $\delta^2 = 1/b$ should fall in the range (0.025, 0.05).

Unless otherwise stated in this exercise the insured population will be considered to have heterogeneity level of $\delta^2 = 0.05$
We use of the Bayesian Inference with Gibbs sampling and Metropolis algorithm to estimate parameters of the proposed baseline distribution.

Bayesian Inference
Bayesian Inference is based on bayes theorem and using probability to quantify for uncertainty in inferences based on statistical data analysis. Gelman *et al.* (1995) [8] describes 3 steps to follow when performing Bayesian data analysis. i.e.
1. Set up a full probability model
2. Condition on the observed data
3. Evaluate the fit of the model based on the resulting posterior distribution.

Applications in Insurance Industry
The aims of this exercise are threefold:
- The first aim is to show that when heterogeneity is disregarded the expected residual lifetime is underestimated.
- Secondly, is that neglecting heterogeneity leads to an underestimation of the insurer's liability.
- Finally, is to show the relevance of the proposed non-central Gamma frailty mixture to reflect an insurer's mortality rating.

Data Analysis and Results
Consider three hypothetical insurers i.e. insurer x, y and z.
Insurer X assumes the population to be homogeneous and applies the Kenyan (KE) 2001-2003 life tables.
Insurer Y assumes the population to be heterogeneous and uses frailty modeling to account for heterogeneity.
Insurer Z carries out underwriting and adjusts the rates to reflect safety loadings. In this case data from Association of Kenya Insurers (AKI) 2007-2010 graduated rates is used.

Case 1: Inverse Gaussian Frailty
The inverse Gaussian frailty mixture is given by:

$$h(t) = \frac{h_o(t)}{(1+2*\delta^2 H_o(t))^{1/2}}$$

The frailty model: $h(t|Z) = Z * h_o(t)$

When $Z = 1$ the hazard corresponds to an average individual, hence $h(t|1) = h_o(t)$. Thus the baseline hazard $h_o(t)$ can be approximated parametrically i.e. $h_o(t)$ ~ Weibull (λ, p)

$$h(x) = \frac{\lambda p x^{p-1}}{(1+2\,\lambda x^p \delta^2)^{1/2}}$$

Case 2: Non-central Gamma Frailty
The inverse Gaussian frailty mixture is given by:

$$h(t) = \frac{h_o(t)}{(1 + 1/2 * \delta^2 H_o(t))^2}$$

Assumption for the baseline is:

$h_o(t)$ ~ Weibull (λ, p)

$$h(t) = \frac{\lambda p x^{p-1}}{(1 + \frac{1}{2} * \delta^2 \lambda x^p)^2}$$

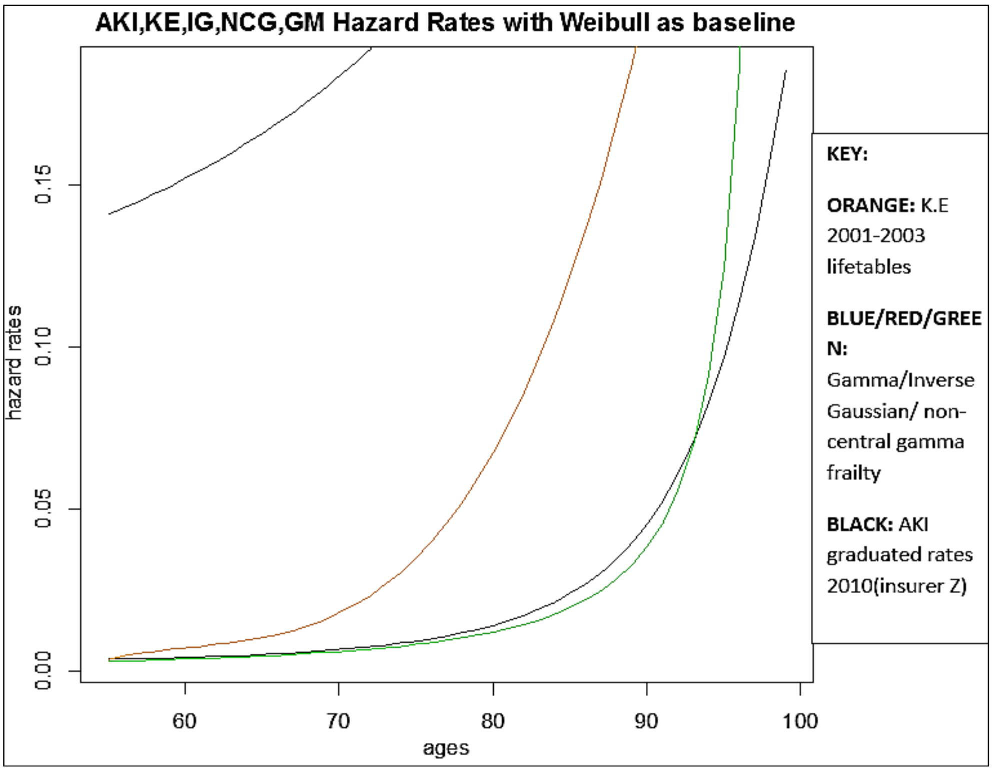
Case 3: Gamma Frailty
The Gamma frailty mixture is given by:

$$h(x) = \frac{h_o(x) b}{b + H_o(x)}$$

Using similar assumptions for the baseline i.e.

$h_o(t)$ ~ Weibull (λ, p)

$$h(x) = \frac{\lambda p x^{p-1} * b}{b + \lambda x^p}$$



Graphs

Results

1. Ignoring heterogeneity leads to underestimation of life expectancy thereby underestimating the expected liability.
2. The non-central gamma closely represents the AKI 2010 graduated rates.

Pension Scheme

Pension schemes are deferred annuities whose benefits are payable on retirement.

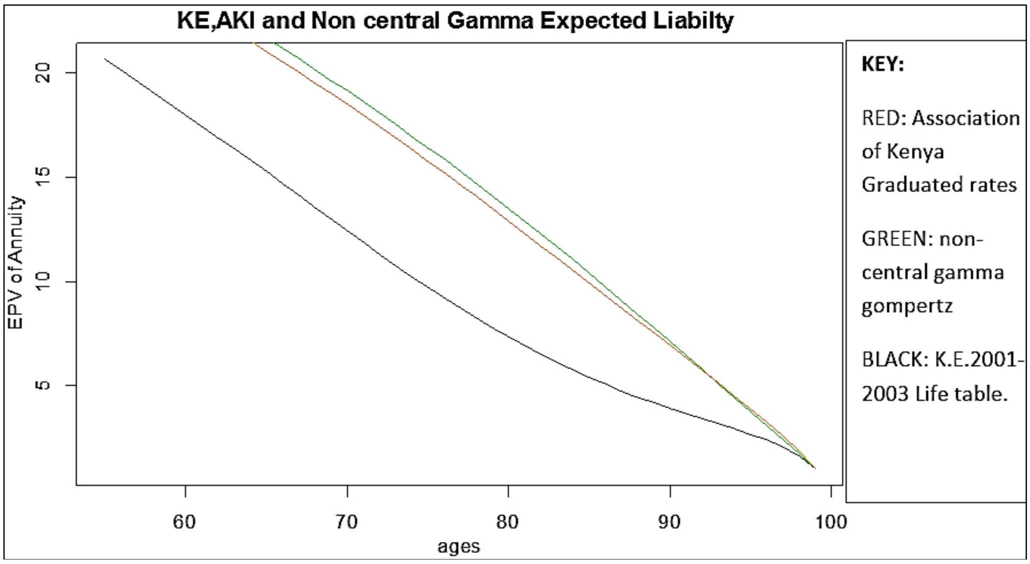
Present Value

These are annuities which commence in m (say) years' time, provided that the annuitant is then active. Thus the present value of amount b payable for a future lifetime $T_{(x+t)}$

$$m|\bar{a}_x = \frac{D_{x+m}}{D_x} * a_{x+m}$$

Where $\frac{D_{x+m}}{D_x}$ is a pure endowment factor and a_{x+m} is an annuity factor at age x+m

For illustration purposes any safety loadings assigned by the insurer is not accounted for since the focus is on the effects of heterogeneity.



Results

1. When heterogeneity is disregarded the expected liability is underestimated.
2. The non-central gamma frailty is a close estimate of the insurer liability. The correlation is also higher.

Conclusion and Recommendation

The conclusion to be reached from the analyses and discussions is that comparing the standard life tables with the Gamma-Weibull and Inverse Gaussian-Weibull model; shows an increase in the insurers expected liability when heterogeneity is considered. That is, assuming the insured to be homogeneous could lead to an underestimation of future liability.

Further, using Non-central Gamma model in estimating future liability by directly adjusting the A.K.I mortality tables shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk.

A key point to note is that the non-central gamma frailty model as proposed gives better estimate of the insurer rates compared to the gamma and inverse gaussian frailty model with similar assumptions of the population level of heterogeneity. The correlation coefficient between the non-central gamma and insurers rates is also higher.

Thus, the non-central family distributions is recommended for further research as it gives better estimates for the insurer's rating.

Appendix

Rcode1: Weibull distribution as baseline

```
Tdata = read.csv ("C:/Users/Dell/Desktop/Book1.csv")
Attach (tdata)
t1=55:99
Alpha = 5.088#bayesian estimates
Beta = 0.7145#bayesian estimates
hx_IG = (alpha*beta*((100-t1) ^ (beta-1)))/sqrt(1+2*alpha*((100-t1)^beta)*0.5)
hx_NCG = (alpha*beta*((100-t1) ^ (beta-1)))/((1+0.5*alpha*((100-t1)^beta)*0.5)^2)
hx_GM = (alpha*beta*((100-t1) ^ (beta-1))*20)/(20+alpha*((100-t1)^beta))
plot (t1,AKI,type="l",xlab="ages",ylab="hazard rates",main="AKI,KE,IG,NCG,GM Hazard Rates with Weibull as baseline")
Lines (t1, KE, col="orange")
Lines (t1, hx_IG,col="blue")
Lines (t1, hx_GM,col="red")
Lines (t1, hx_NCG,col="green")
```

Rcode 2: Insurers Expected Liability

```
Ttdata = read.csv ("C:/Users/Dell/Desktop/Book3.csv")
Attach (ttdata)
t1=55:99
Plot (t1, KE, xlab="ages", ylab="EPV of Annuity", main="KE, AKI and Non central Gamma Expected Liabilty ", type="l")
Lines (t1, NCG, col="green")
Lines (t1, AKI, col="orange")
```

Rcode3: Pearson's Correlation test between the AKI and KE vs AKI and NCG

```
> cor.test (KE, AKI)
Pearson's product-moment correlation
Data: KE and AKI
t = 35.566, df = 43, p-value < 2.2e-16
Alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
0.9698573 0.9909129
Sample estimates:
Cor
```

0.9834241

```
> cor.test (NCG, AKI)
Pearson's product-moment correlation
Data: NCG and AKI
t = 345.76, df = 43, p-value < 2.2e-16
Alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
0.9996708 0.9999018
Sample estimates:
Cor
0.9998202
```

Win BUGS MCMC Weibull distribution parameter estimation

```
MODEL<-function () {
For (i in 1: N) {
```

s[i] ~ dweib (beta, alpha) }
Beta ~ dexp (0.0000009)
Alpha ~ dexp (0.9)}
Write. Model (MODEL, "MODEL.txt")
INIT<-Function () {list (beta=dexp (0.0000009), alpha=dexp (0.9))}
DATA=list(s=c(0.003667,0.004641,0.005445,0.006119,0.006692,0.007206,0.00771,0.008224,0.008809,0.009485,0.010303,0.011304,0.012518,0.013987,0.015753,0.017859,0.020346,0.023258,0.026631,0.030511,0.034954,0.039989,0.045688,0.052083,0.059251,0.067219,0.076071,0.085863,0.096676,0.108566,0.121625,0.135946,0.151614,0.16875,0.187463,0.207886,0.230168,0.254479,0.280998, 0.309955,0.341603,0.376222,0.414138,0.455785,0.501635,0.552255), N=45)
BUGS=bugs(data=DATA,inits=INIT,parameters.to.save=c("beta","alpha"),model.file="MODEL.txt",bugs.directory="C:/Program Files/R/R-3.5.3/library/R2WinBUGS",n.chains=1,n.iter=10000,n.burnin=100,codaPkg=TRUE,debug=T)
Node statistics

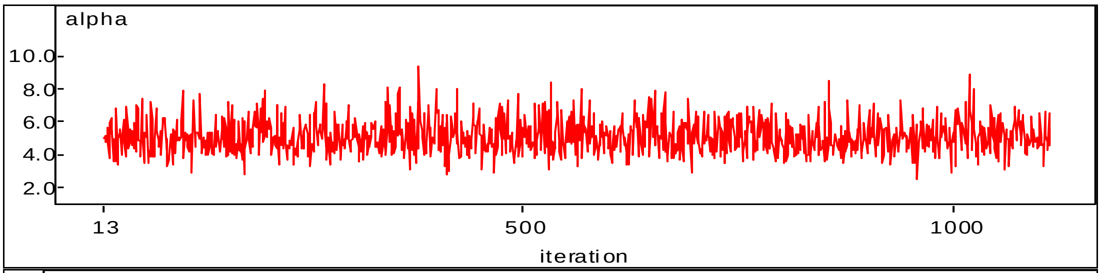
Node	mean	sd	MCerror	2.5%	median	97.5%	start	Sample
Alpha	5.108	1.029	0.02967	3.417	5.003	7.325	13	1100
Beta	0.7127	0.08163	0.002268	.5615	0.7091	0.8866	13	1100
Deviance	-111.0	2.239	0.05732	-113.4	-111.6	-104.9	13	1100

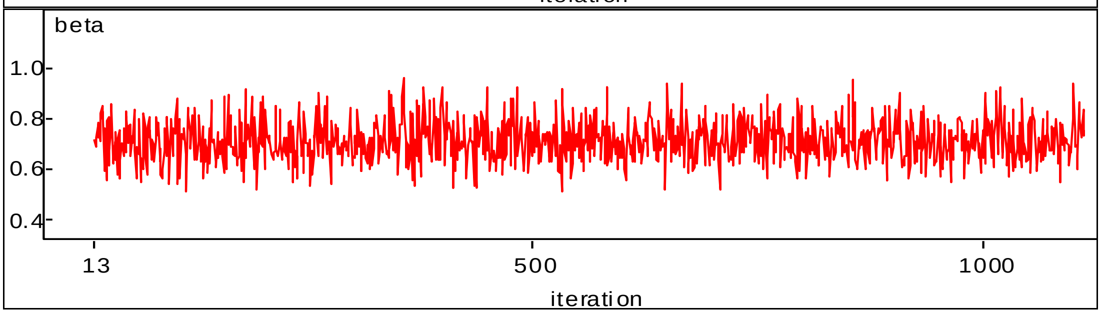
dic.stats()
DIC

Dbar = post.mean of -2logL; Dhat = -2LogL at post.mean of stochastic nodes

	Dbar	Dhat	pD	DIC
S	-111.001	-112.939	1.938	-109.062
Total	-111.001	-112.939	1.938	-109.062

History (*, C:/Users/Dell/AppData/Local/Temp/RtmpczALiA/history.odc)





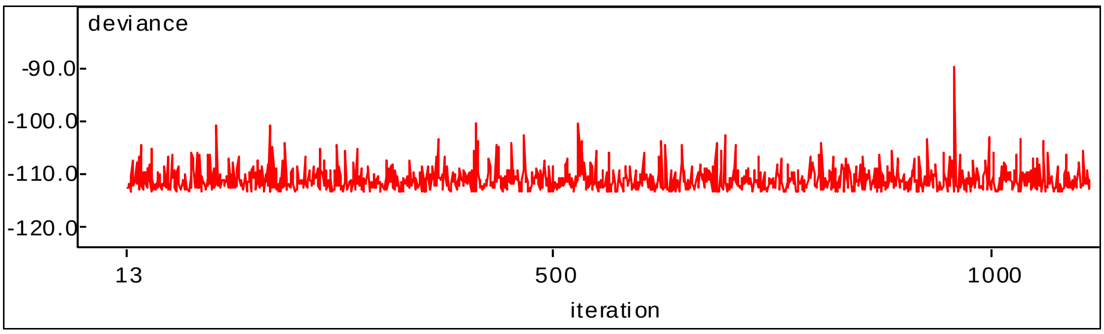


Table 1: Hazard Rates

	KE Lifetable	AKI 2010 Male Graduated Rates	Gamma-Weibull Frailty	Inverse Gaussian-Weibull Frailty	Non Central Gamma- Weibull Frailty
Age (x)	Mx	µx	µx	Mx	µx
55	0.003667	0.003739	0.2522328	0.1386372	0.0029735
56	0.004641	0.00377	0.2571093	0.1406397	0.0030854
57	0.005445	0.003856	0.2621840	0.1427179	0.0032042
58	0.006119	0.003964	0.2674693	0.1448766	0.0033303
59	0.006692	0.004096	0.2729790	0.1471209	0.0034646
60	0.007206	0.00424	0.2787280	0.1494564	0.0036077
61	0.00771	0.004394	0.2847324	0.1518891	0.0037604
62	0.008224	0.004557	0.2910102	0.1544258	0.0039237
63	0.008809	0.004732	0.2975807	0.1570739	0.0040985
64	0.009485	0.004922	0.3044655	0.1598414	0.0042861
65	0.010303	0.005134	0.3116881	0.1627372	0.0044878
66	0.011304	0.005371	0.3192745	0.1657713	0.0047049
67	0.012518	0.005639	0.3272538	0.1689545	0.0049393
68	0.013987	0.005939	0.3356578	0.1722989	0.0051929
69	0.015753	0.006275	0.3445223	0.1758183	0.0054678
70	0.017859	0.006648	0.3538873	0.1795277	0.0057667
71	0.020346	0.007066	0.3637974	0.1834444	0.0060925
72	0.023258	0.007533	0.3743030	0.1875875	0.0064486
73	0.026631	0.008055	0.3854609	0.1919790	0.0068391
74	0.030511	0.00864	0.3973357	0.1966436	0.0072686
75	0.034954	0.009297	0.4100007	0.2016100	0.0077428
76	0.039989	0.010038	0.4235401	0.2069108	0.0082684
77	0.045688	0.010875	0.4380507	0.2125837	0.0088532
78	0.052083	0.011824	0.4536442	0.2186729	0.0095068
79	0.059251	0.012904	0.4704509	0.2252296	0.0102410
80	0.067219	0.014139	0.4886234	0.2323148	0.0110701
81	0.076071	0.015556	0.5083416	0.2400004	0.0120117
82	0.085863	0.017189	0.5298193	0.2483729	0.0130880
83	0.096676	0.019078	0.5533131	0.2575370	0.0143271
84	0.108566	0.021274	0.5791335	0.2676205	0.0157648
85	0.121625	0.023837	0.6076606	0.2787819	0.0174479
86	0.135946	0.02684	0.6393648	0.2912197	0.0194380
87	0.151614	0.030376	0.6748368	0.3051869	0.0218179
88	0.16875	0.034552	0.7148296	0.3210110	0.0247013
89	0.187463	0.039504	0.7603199	0.3391253	0.0282479
90	0.207886	0.045393	0.8126024	0.3601159	0.0326880
91	0.230168	0.052411	0.8734345	0.3847985	0.0383654
92	0.254479	0.060792	0.9452748	0.4143468	0.0458127
93	0.280998	0.070806	1.0316893	0.4505204	0.0558955
94	0.309955	0.082769	1.1380929	0.4960988	0.0701064
95	0.341603	0.097035	1.2732146	0.5557793	0.0912233
96	0.376222	0.113999	1.4523188	0.6382683	0.1249790
97	0.414138	0.134074	1.7054555	0.7620134	0.1850835
98	0.455785	0.157692	2.1042529	0.9754911	0.3129418
99	0.501635	0.185265	2.8980995	1.4733706	0.7042594

Table 2: Non Central Gamma Annuity Function I=2%

Age (x)	l_x	d_x	p_x	q_x	μ_x	D_x	N_x	a_x
55	100000	297	0.997031	0.002969	0.002974	33650.42	881781.6	26.20417
56	99703	307	0.996919	0.003081	0.003085	32892.66	848131.2	25.78481
57	99396	318	0.996801	0.003199	0.003204	32148.36	815238.5	25.35863
58	99078	329	0.996675	0.003325	0.00333	31417.17	783090.1	24.92554
59	98749	342	0.996541	0.003459	0.003465	30698.75	751673	24.48546
60	98407	354	0.996399	0.003601	0.003608	29992.72	720974.2	24.03831
61	98053	368	0.996247	0.003753	0.00376	29298.73	690981.5	23.58401
62	97685	383	0.996084	0.003916	0.003924	28616.43	661682.8	23.12247
63	97302	398	0.99591	0.00409	0.004099	27945.46	633066.3	22.65364
64	96904	414	0.995723	0.004277	0.004286	27285.45	605120.9	22.17742
65	96490	432	0.995522	0.004478	0.004488	26636.04	577835.4	21.69375
66	96058	451	0.995306	0.004694	0.004705	25996.83	551199.4	21.20256
67	95607	471	0.995073	0.004927	0.004939	25367.46	525202.6	20.70379
68	95136	493	0.994821	0.005179	0.005193	24747.52	499835.1	20.19738
69	94643	516	0.994547	0.005453	0.005468	24136.61	475087.6	19.68328
70	94127	541	0.99425	0.00575	0.005767	23534.31	450951	19.16143

71	93586	568	0.993926	0.006074	0.006093	22940.18	427416.7	18.63179
72	93017	598	0.993572	0.006428	0.006449	22353.76	404476.5	18.09433
73	92419	630	0.993184	0.006816	0.006839	21774.59	382122.7	17.54902
74	91789	665	0.992758	0.007242	0.007269	21202.13	360348.1	16.99584
75	91125	703	0.992287	0.007713	0.007743	20635.86	339146	16.43479
76	90422	745	0.991766	0.008234	0.008268	20075.2	318510.1	15.86585
77	89677	790	0.991186	0.008814	0.008853	19519.5	298435	15.28907
78	88887	841	0.990538	0.009462	0.009507	18968.09	278915.4	14.70445
79	88046	897	0.989811	0.010189	0.010241	18420.22	259947.4	14.11207
80	87149	959	0.988991	0.011009	0.01107	17875.04	241527.1	13.51198
81	86189	1029	0.98806	0.01194	0.012012	17331.62	223652.1	12.90428
82	85160	1107	0.986997	0.013003	0.013088	16788.9	206320.5	12.2891
83	84053	1196	0.985775	0.014225	0.014327	16245.69	189531.6	11.66658
84	82857	1296	0.984359	0.015641	0.015765	15700.58	173285.9	11.03691
85	81561	1411	0.982703	0.017297	0.017448	15151.97	157585.3	10.40032
86	80150	1543	0.98075	0.01925	0.019438	14597.93	142433.3	9.757091
87	78608	1696	0.978418	0.021582	0.021818	14036.19	127835.4	9.107556
88	76911	1877	0.975601	0.024399	0.024701	13463.99	113799.2	8.452117
89	75035	2090	0.972147	0.027853	0.028248	12877.93	100335.2	7.791256
90	72945	2346	0.96784	0.03216	0.032688	12273.77	87457.3	7.125547
91	70599	2657	0.962361	0.037639	0.038365	11646.13	75183.53	6.455669
92	67942	3042	0.955221	0.044779	0.045813	10988.02	63537.41	5.782426
93	64899	3528	0.945638	0.054362	0.055896	10290.18	52549.39	5.10675
94	61371	4155	0.932295	0.067705	0.070106	9539.987	42259.2	4.429692
95	57216	4988	0.912814	0.087186	0.091223	8719.685	32719.22	3.75234
96	52228	6136	0.882515	0.117485	0.124979	7803.382	23999.53	3.07553
97	46092	7788	0.831035	0.168965	0.185084	6751.573	16196.15	2.398871
98	38304	10292	0.731292	0.268708	0.312942	5500.777	9444.578	1.716953
99	28011	28011	0.494475	0.505525	0.704259	3943.801	3943.801	1

Table 3: Annuity Functions i=2%

	KE Lifetable-Males	**AKI graduated annuity**	**Non central Gamma frailty**
Age (x)	**a_x**	**a_x**	**a_x**
55	20.64573	25.55944	26.20417
56	20.11226	25.14447	25.78481
57	19.58518	24.72038	25.35863
58	19.06039	24.28827	24.92554
59	18.53466	23.84838	24.48546
60	18.00544	23.40101	24.03831
61	17.471	22.94612	23.58401
62	16.93044	22.48361	23.12247
63	16.38323	22.01338	22.65364
64	15.82972	21.53531	22.17742
65	15.27047	21.04937	21.69375
66	14.70662	20.55562	21.20256
67	14.13968	20.05417	20.70379
68	13.5713	19.54515	20.19738
69	13.00334	19.02872	19.68328
70	12.43781	18.50505	19.16143
71	11.87679	17.97425	18.63179
72	11.32236	17.4365	18.09433
73	10.77656	16.89201	17.54902
74	10.24123	16.34095	16.99584
75	9.718082	15.78355	16.43479
76	9.208765	15.22007	15.86585
77	8.714551	14.6508	15.28907
78	8.236692	14.07607	14.70445
79	7.776061	13.49622	14.11207
80	7.333477	12.91169	13.51198
81	6.909322	12.32294	12.90428
82	6.503921	11.73047	12.2891
83	6.117333	11.13484	11.66658
84	5.749496	10.53664	11.03691
85	5.400042	9.936531	10.40032
86	5.068486	9.335151	9.757091
87	4.754157	8.733133	9.107556
88	4.456128	8.131069	8.452117
89	4.173277	7.529404	7.791256
90	3.904111	6.928355	7.125547

91	3.646674	6.327735	6.455669
92	3.398298	5.7267	5.782426
93	3.155166	5.123419	5.10675
94	2.9115	4.514484	4.429692
95	2.658191	3.894106	3.75234
96	2.380078	3.252793	3.07553
97	2.050665	2.575316	2.398871
98	1.62152	1.837366	1.716953
99	1	1	1

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