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# The relationships of partial derivatives, continuity and total differentials of multivariate functions

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**Abstract**
On the basis of discussion of the properties of Multivariate functions, this article puts forward the relationships of partial derivatives, continuity and total differentials of multivariate functions.

**Keywords:** Multivariate functions, partial derivatives, continuity, total differentials

## 1. Introduction
It is well known that if a univariate function is differentiable, then it is continuous [1]. Is this situation the same when the function is multivariate? Namely, if the partial derivatives of a multivariate function exist, is the function continuous? The answer is negative. Here we will explain it with the function of two variables in detail.

## 2. The relationships of partial derivatives and continuity
If a function of two variables is continuous at a point, the partial derivatives of this point does not necessarily exist.
Example 1 [2]. Let $f(x,y) = |x| + |y|$, then it is continuous at (0,0), and its partial derivatives at this point do not exist.
Proof. It is easy to know that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0)$

$\therefore f(x,y)$ is continuous at (0,0).

$$\because\ f_x(0,0) = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}$$

$\therefore f_x(0,0)$ Does not exist.
Similarly $f_y(0,0)$ does not exist.
$\therefore f(x,y)$
is continuous at(0,0), but its partial derivatives at this point do not exist.

If the partial derivatives of a function of two variables exist at a point, this function is not necessarily continuous at this point.
Example 2 [3]. Let $f(x,y) = \begin{cases} 0, & x = 0 \ or \ y = 0 \\ 1, & others \end{cases}$,
Then its partial derivatives exist at(0,0), it is not continuous at this point.

Proof. $f_x(0,0) = \lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0$

; Similarly $\boldsymbol{f_y(0,0) = 0}$

$$\because \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=1\neq f(0,0)$$

; $\therefore f(x,y)$ is not continuous at$(0,0)$.
$\therefore$The partial derivatives of $f(x,y)$ exist at $(0,0)$, the function is not continuous at this point.

### 3. The relationships of partial derivatives and total differential

If the partial derivatives of a function of two variables exist at a point, the function is not necessarily differentiable at this point. Firstly, the function with total differential is defined to be differentiable [3][4].

Example 3 [3]. Let$f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & x^2+y^2=0\\ 0, & x^2+y^2\neq 0\end{cases}$,

Then its partial derivatives exist at point （0,0）, it is not differentiable at this point.

Proof. $\because f_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

So its partial derivatives exist at point （0,0）.

$$\because \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\cdot\Delta x-f_y(0,0)\cdot\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x)^2\cdot\Delta y}{[(\Delta x)^2+(\Delta y)^2]}$$

When $(x,y)$ approaches $(0,0)$ by the path $y=kx$

$$\lim_{\substack{y=kx\\ x\to 0}}\frac{x^2\cdot y}{(x^2+y^2)^{\frac{3}{2}}}=\lim_{x\to 0}\frac{kx^3}{(1+k^2)^{\frac{3}{2}}\cdot x^3}=\frac{k}{(1+k^2)^{\frac{3}{2}}}$$

Obviously, the limit varies according to the value of $k$, so it does not exist. Which means that $f(x,y)$ is not differentiable at （0,0）.

If the partial derivatives of a function of two variables exist and they are continuous at a point, then the function is differentiable at this point.
This conclusion can be proved easily [3].
At a certain point, if a function of two variables is differentiable, and its partial derivatives exist, but they are not necessarily continuous.

Example 4 [3]. Let$f(x,y)=\begin{cases}(x^2+y^2)\,sin\frac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$,

then its partial derivatives exist, but they are not necessarily continuous.

Proof. $\because f_x(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\Delta x\cdot\sin\dfrac{1}{\Delta x^2+\Delta y^2}=0$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\Delta y\cdot\sin\frac{1}{\Delta x^2+\Delta y^2}=0$$

$$\therefore \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\cdot\Delta x-f_y(0,0)\cdot\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\sqrt{\Delta x^2+\Delta y^2}\cdot\sin\frac{1}{\Delta x^2+\Delta y^2}=0$$

$\therefore f(x,y)$ is not differentiable at$(0,0)$
And consider the situation when $x^2+y^2\neq 0$

$$f_x(x,y) = 2 \cdot sin\frac{1}{x^2+y^2} + (x^2+y^2) \cdot cos\frac{1}{x^2+y^2} \cdot \frac{-2x}{(x^2+y^2)^2}$$

$$= 2 \cdot sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2} \cdot cos\frac{1}{x^2+y^2}$$

$$f_y(x,y) = 2 \cdot sin\frac{1}{x^2+y^2} - \frac{2y}{x^2+y^2} \cdot cos\frac{1}{x^2+y^2}$$

Because $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{2x}{x^2+y^2} \cdot \cos\frac{1}{x^2+y^2}$ does not exist, namely $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_x(x,y)$ does not exist, so $f_x(x,y)$ does not continuous at$(0,0)$.

Similarly $f_y(x,y)$ does not continuous at $(0,0)$.

**4. The relationships of total differential and continuity**

1. If a function of two variables is differentiable at a point, it is must continuous at this point.
   It is easy to be proved based on the definition of total differential.
2. If a function of two variables is continuous at a point, it is not necessarily differentiable at this point.
   It is obvious according to example 1. It has been proved that the partial derivatives of this function at $(0,0)$ do not exist, so it is not differentiable at this point.

**5. Conclusion**

In summary, the relationships of partial derivatives, continuity and total differentials of multivariate function can be concluded as follows:

For multivariate functions at a certain point, (1) The continuity is the unnecessary and insufficient condition of the existence of the partial derivatives; (2) The existence of the partial derivatives is the necessary and insufficient condition of the differentiability. (3) The continuity of the partial derivatives is the unnecessary and sufficient condition of the differentiability; (4) The differentiability is the unnecessary and sufficient condition of the continuity.

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