International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2019; 4(4): 66-68

www.mathsjournal.com
Received: 27-05-2019
Accepted: 29-06-2019

Ashok Kumar Pandey
Associate Prof. Ewing Christian College (An Autonomous Post graduate College of Allahabad University, Prayagraj), Prayagraj, Uttar Pradesh, India

Correspondence
Ashok Kumar Pandey
Associate Prof. Ewing Christian College (An Autonomous Post graduate College of Allahabad University, Prayagraj), Prayagraj, Uttar Pradesh, India

Prime ideals and radical of ideals in near- rings

Ashok Kumar Pandey

Abstract
The most interesting and active field of current research in mathematics is theory of near- ring due to its wide application in coding theory, group theory, geometry, cryptography and block- designing etc. Near rings are one of the generalized structure of rings. In this paper we discuss the concepts of ideals and radicals in near- rings also the concepts of m- system and sp- system in context of near- rings. Also, the concepts of m^*- system, m^*- sequences and prime radicals in near rings.

Keywords: Near- ring, prime ideal, semi- prime ideal, radical of ideal, m- system, sp- system, m^*- system.

Introduction
The most interesting and active field of current research in mathematics is theory of near- ring due to its wide application in coding theory, group theory, geometry, cryptography and block- designing etc. Near rings are one of the generalized structure of rings, The study and research on near- rings is very systematic and continuous. Near- rings around in all directions of mathematics and continuous research is being conducted, which show that their structure has power and beauty all its own. Modern ring theory has very active mathematical discipline and studying rings in their own right. In 1930 Wieland studied near- rings. In this paper we discuss the concepts of ideals and radicals in near- rings also the concepts of m- system and sp- system in context of near- rings. Also, the concepts of m^*- system, m^*- sequences and prime radicals in near rings.

Definition 1.1: A system $(N, +, .)$ is called a near- ring if
a) $(N, +)$is a group (not necessarily abelian).
b) $(N, .)$ is a semi group
c) Multiplication is either left distributive or right distributive over addition. That is $(n_1 + n_2).n_3 = n_1 .n_3 + n_2 .n_3$,
Or, $n_1.(n_2 + n_3) = n_1 .n_2 + n_1 .n_3$, for $all\ n_1, n_2, n_3 \in N$.

Example 1: Every ring is a near- ring.

Definition 1.2: A subset S of a near- ring N is called an m- system if for every $a, b \in S$, there exist $a_1 \in < a >$ and $b_1 \in < b >$ such that a_1, $b_1 \in S$.

Example 2: Let M= $\mathbb{Z}_4 = \{0, 1, 2, 3\} mod(4), S = \{1, 2\} mod(4)$. Clearly, $S \subseteq \mathbb{Z}_{4,}$ S is an m- system.

Theorem 1.3: An ideal P in N is a prime ideal if and only if its complement P^C is an $m -$ system.

Proof: Suppose that P is a prime ideal then to prove that P^C is an m- system.
Let $a, b \in P^C \Rightarrow < a > \subseteq P^C\ and < b > \subseteq P^C$
$\Rightarrow \exists\ a_1 \in < a > \subseteq P^C\ and\ b_1 \in < b > \subseteq P^C \Rightarrow < a > \nsubseteq P\ and\ < b > \nsubseteq P$
Hence by the definition of prime ideal we have, $< a >< b > \nsubseteq P$.

So, $a_1b_1 \notin P \Rightarrow a_1b_1 \in P^C$. Hence P^C is an m- system.
Conversely, suppose that P^C is an m- system, then to prove that P is a prime ideal, Suppose that B and D be two ideals such that $BD \subseteq P$ then to prove either $B \subseteq P$ or $D \subseteq P$. Suppose that neither $B \subseteq P$ nor $D \subseteq P$.
$\Rightarrow B \subseteq P^C$ or $D \subseteq P^C$ then as P^C is an m- system, so for every $b \in B \subseteq P^C$ and $d \in D \subseteq P^C \exists b' \in < b > \subseteq B$ and $d' \in < d > \subseteq D$ such that $b'd' \in P^C$
$\Rightarrow b'd' \notin P$ so $BD \nsubseteq P \Rightarrow BD \subseteq P^C$,
which is a contradiction. Hence P is a prime ideal.

Definition 1.4: An ideal S of N is said to be a semi- prime ideal if for any ideal J of N, $J^2 \subseteq S \Rightarrow J \subseteq S$.

Definition 1.5: A subset S of a near- ring N is said to be an sp- system if for any $s \in S$, there exist $s_1, s_2 \in < s >$ such that $s_1 . s_2 \in S$.

Example 3: For any $n \in N$, the set $\{n, n^2, n^3, \dots \dots\}$ is an sp- system.

Theorem 1.6: P be a semi- prime ideal in a near- ring N if and only if $N \backslash P$ is an $sp -$ system in N.

Proof: Suppose that P is a semi- prime ideal of N. Then for any ideal J of N such that $J^2 \subseteq P \Rightarrow J \subseteq P$.

Claim: $N \backslash P$ is an $sp -$ system in N.
Let $m \in N \backslash P$, then $< m > \subseteq N \backslash P$. Let $m', m'' \in < m > \subseteq N \backslash P$ then to show $m'.m'' \in < m > \subseteq N \backslash P$. Now $< m > \subseteq N \backslash P \Rightarrow < m > \nsubseteq P$.
So by definition of semi- prime ideal it follows that $< m >^2 \nsubseteq P$, hence $m'.m'' \notin P \Rightarrow m'.m'' \in < m > \subseteq N \backslash P$. Thus $N \backslash P$ is an sp- system.
Conversely, let $N \backslash P$ be an sp- system that is for all $n \in N \backslash P, \exists n_1, n_2 \in < n >$ such that $n_1.n_2 \in N \backslash P$.

Claim: P is a semi- prime ideal.
Let J be an ideal of N such that $J^2 \subseteq P$, we need to show $J \subseteq P$. If it possible to show $J \nsubseteq P$, then $J \subseteq N \backslash P$, which is an sp- system. Therefore, $\forall n \in N \backslash P, \exists n_1, n_2 \in < n >$ such that $n_1.n_2 \in N \backslash P$.
Hence $n_1.n_2 \notin P \Rightarrow J^2 \nsubseteq P$,
which is a contradiction. Hence P is a semi- prime ideal.

Theorem 1.7: If I is any semi- prime ideal in a near ring N, then I is the intersection of all minimal prime ideals of I in N.

Proposition 1.8: If I is a semi- prime ideal in a near ring N, then I is the intersection of all the prime ideals containing I.

Theorem 1.9: An ideal I in a near ring N is a semi- prime ideal in N if and only if $\wp(I) = I$.

Proof: Suppose that $\wp(I) = I$, then to prove the ideal I is a semi- prime ideal.
Since $\wp(I)$ is the intersection of all prime ideals containing I and the intersection of all prime ideals is a semi- prime ideal, so that I is semi- prime ideal.
Conversely, suppose that I is semi- prime ideal, then to show $\wp(I) = I$.
Already we know that $I \subseteq \wp(I)$. Now if it possible, suppose that $\wp(I) \nsubseteq I$, then there exists $a \in \wp(I)$ such that $a \notin I$. Since I is a semi- prime ideal, so we have $N \backslash I$ is an sp- system. Hence there exists an m- system M in N such that $a \in M \subseteq N \backslash I$. Now $a \in \wp(I)$ and we know every m- system which contains I. But $I \cap N \backslash I = \emptyset$, and therefore $I \cap M$ is empty, which is a contradiction. Hence, $\wp(I) = I$.

2. Radicals of an ideal in near- rings

Definition: A subset M of N is said to an m^*- system, if for any $a, b \in M$, there exists an element $b_1 \in (b)$ such that $ab_1 \in M$.
Note: It is a generalization of m-system.

Theorem 2.1: If M is a non empty m^*- system and if I is an ideal of N such that $I \cap M = \emptyset$, then there exists a prime ideal P containing I such that $P \cap M = \emptyset$.

Proof: Suppose M is a non empty m^*- system and if I is an ideal of N such that $I \cap M = \emptyset$. We Consider N' = $\{K |\ K$ is an ideal of, $I \subseteq K, K \cap M = \emptyset\}$. We want to show N' contains a maximal element. Clearly N' is a partially ordered set with respect to the relation "$\subseteq$" and since $I \in N'$ so $N' \neq \emptyset$. We show that every totally ordered subset T of N' has an upper bound in N'.
Let $T_0 = \cup T_i$ where $T_i \in T$, then obviously T_0 is an upper bound of N'. Now to show $T_0 \in N'$, since $T_i \in T \in N'$. So $I \subseteq T_i\ \forall i$. This show that $I \subseteq T_0$, if $x \in T_0$, then $x \in T_i$ for some $T_i \in T$. But T_i is an ideal of N, so $xa \in T_i \in T_0$ and $a(b + x) - ab \in T_i \in T_0$ for all $x \in T_i$ and $a, b \in N$. This shows that T_0 is an ideal of N. Also, if $T_0 \cap M \neq \emptyset$, then there exists $x \in M$ such that $x \in T_0 = \cup T_i$. This implies that $x \in T_i$, for some $T_i \in T$. Thus $T_i \cap M \neq \emptyset$ for some i, which is a contradiction as $T_i \cap M \neq \emptyset$. This shows that $T_i \cap M = \emptyset$. So, $T_0 \in N'$, then by using the Zorn's Lemma N' has a maximal element say 'P'. Since $N \backslash P$ is an m- system then P is a prime ideal.

Definition 2.2: A sequence $s_{0,}s_{1,}\ s_{2,}\ldots\ldots\ldots\ldots s_{n,}$ of elements in N is called an m^*- sequence if $s_{n,} \in s_{n-1} < s_{n-1,} >, \forall\ n \geq 1$.

Theorem 2.3: If I be a proper ideal of N, then $\wp(I)$= $\{a \in N \mid every\ m^* -$ sequence $\{a_n\}n \geq 0$ starting with $a \in N$ meets $I\}$.

Definition 2.4: An element a in a near- ring N is said to be strongly nilpotent if every $m^* -$ sequence $\{a_n\}n \geq 0$ with $a_{0,} = a$ vanishes (that is there is a positive integer$n$ such that $a_{k,} = 0$ for all $k \geq n$).

Theorem 2.5: Every strongly nilpotent element in N is nilpotent but converse is not true.

Example: consider the ring of all $3X3$ matrices over the ring of integers.

Let $a_{0,} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\ and\ x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Define $a_1 = a_1 x a_0,\ \ldots\ldots a_n = a_{n-1} x\ a_{n-1}$. Now$a_n \geq 0$ for all $n \geq 0$ and hence $a_{0,}$ is not strongly nilpotent element. But $(\ a_{0,})^3 = 0$. This show that a_0 is nilpotent.

Definition 2.6: For a proper ideal I of N, the prime radical of I is the intersection of all prime ideals containing I and it is denoted by $\boldsymbol{\wp}(\boldsymbol{I})$. The prime radical of N is defined as $\boldsymbol{\wp}(\{\mathbf{0}\})$.

Proposition 2.7: The prime radical of N is the set of all strongly nilpotent elements in N.

Theorem 2.8: Let$I$ be an ideal of N, then $\wp(I)$ = $\{a \in N \mid every\ m^*$ $-$system that contains a must contain an element of $I\}$.

Proof: Let X be a right hand side set. Let $x \in\ \boldsymbol{\wp}(\boldsymbol{I})$.If $x \notin X$, then there exists an m^* $-$system M with $x \in M$ and $M \cap I =\ \emptyset$. This implies there exists a prime ideal P such that $P \supseteq I$ and $M \cap P =\ \emptyset$. So, $x \notin P$. Therefore, $x \notin \boldsymbol{\wp}(\boldsymbol{I})$, a contradiction.
Conversely, suppose $x \in X$. If $x \notin \boldsymbol{\wp}(\boldsymbol{I})$, then there exists a prime ideal P containing I such that $x \notin P$. This implies $x \in N \backslash P$ and $N \backslash P$ is an m^* $-$system.Since $x \in X$, we have $N \backslash P \cap I \neq\ \emptyset$, is a contradiction.

Theorem 2.9: Let $\alpha_{1,}\alpha_{2,}\alpha_{3,\ldots\ldots\ldots,}\alpha_{n,} \subseteq B$ ideals of N. Then $\alpha_{1,}\alpha_{2,}\alpha_{3,\ldots\ldots\ldots,}\alpha_{n,} \subseteq B$, if and only if $B_1 B_2 \ldots . B_n \subseteq B$ Where $B_i =\ \alpha_i + B = \{x_i + b_i | x_i \epsilon\ \alpha_i, b_i \in B\}$ for $i\ =\ 1, 2, 3, \ldots\ldots, n$.

Proof: Since the sum of two ideals is an ideal each B_i is an ideal. Let y be any element of$B_1 B_2 \ldots . B_n \subseteq B$ then y = $(x_1 + b_1).(x_2 + b_2). \ldots . (x_n + b_n)$,
Where $x_i \epsilon\ \alpha_i, and\ b_i \in B\ for\ i = 1, 2, 3, \ldots . , n$.
Now $(x_1 + b_1)(x_2 + b_2) = (x_1 + b_1)(x_2) + ((x_1 + b_1)(b_2))$
$= (x_1 + b_1 - x_1 +\ x_1)(x_2) + ((x_1 + b_1)(b_2))$
$=\ x_1 x_2 + c_1 + (x_1 + b_1)(b_2)$ where $c_1 \in B$.
Therefore, $(x_1 + b_1)(x_2 + b_2) =\ x_1 x_2 +\ b_1$.
Now repeating the above argument, it can be shown that $y =\ x_1 x_2 x_3 x_4 \ldots x_{n-1} x_n +\ b$, where $b \in B$.
k_iTherefore, $y \in B$, if and only if $y =\ x_1 x_2 x_3 x_4 \ldots x_{n-1} x_n \in B$ and hence, $B_1 B_2 \ldots . B_n \subseteq B$if and only if $\alpha_1\ \alpha_2\ \alpha_{3,\ldots\ldots\ldots,}\alpha_{n,} \subseteq B$.

Theorem 2.10: If B is an ideal of N and $b_{0,}\ b_{1,}\ b_{2,}\ldots\ldots\ldots\ldots, b_{n,} \ldots\ldots$is an m- sequence not meeting B, then there exists a prime ideal containing B and not meeting the m- sequence $b_{0,}b_{1,}\ b_{2,}\ldots\ldots\ldots\ldots b_{n,} \ldots\ldots$

Proof: We consider the collection S of all ideals containing B and not meeting the m- sequence $b_{0,}\ b_{1,}\ b_{2,}\ldots\ldots\ldots\ldots b_n$. Since $B \in S, S$ is not empty. By Zorn's lemma, there exists a maximal ideal P in this collection. We want to show that P is a prime ideal. Let $\alpha_{1,}\alpha_{2,}\alpha_{3,\ldots\ldots\ldots,}\alpha_n \subseteq B$ be ideals of N such that $\alpha_{i,} \nsubseteq$ P, $\mathrm{i} = 1, 2, 3, \ldots\ldots\ldots, \mathrm{n}$. Now the ideal $B_i =\ \alpha_{i,} + \mathrm{P}$ contain P properly. Therefore, each B_i meets the m- sequence. Therefore, for each i, there exists a $b_{k_i} \in \mathrm{B_i}$, then every b_r, $\mathrm{r} \geq \mathrm{k_i}$ belongs to $\mathrm{B_i}$ and hence we assume that $\mathrm{k_1} < \mathrm{k_2} < \cdots \ldots . . < \mathrm{k_n}$ and $\mathrm{k_1}, \mathrm{k_2}, \ldots\ldots\ldots, \mathrm{k_n}$ are consecutive integers, Since $b_{0,}\ b_{1,}\ b_{2,}\ldots\ldots\ldots\ldots b_{n,} \ldots$ is an $m -$ sequence, there exists an another sequence $c_{0,}\ c_{1,}\ c_{2,}\ldots\ldots\ldots\ldots c_{n,}$ such that $b_i = b_{i-1} c_{i-1} b_{i-1}$ for $\mathrm{i} = 1, 2, 3, \ldots\ldots\ldots, \mathrm{n}$.
Now $(b_{k_1} c_{k_1} b_{k_1})(c_{k_2} b_{k_2})(c_{k_3} b_{k_3}) \ldots . (c_{k_{n-1}} b_{k_{n-1}})(\mathrm{b}_{k_n} c_{k_n} b_{k_n}) =\ b_{k_{n+1}}$. Since $(b_{k_1} c_{k_1} b_{k_1}) \in \mathrm{B_1}$, $(c_{k_i} b_{k_i}) \in \mathrm{B_i}$ or $\mathrm{i} = 1, 2, 3, \ldots\ldots\ldots, \mathrm{n}$,
Since, $b_{k_{n+1}} \in B_1 B_2 \ldots . B_n$. Since the $m -$ sequence does not meet P, it follows that $B_1 B_2 \ldots . B_n \nsubseteq$ P. Therefore, it follows that $\alpha_{1,}\alpha_{2,}\alpha_{3,\ldots\ldots\ldots,}\alpha_{n,} \nsubseteq$ P and hence P is a prime ideal.

References

1. Baer R. Radical ideals, Amer. J. Math. 1943; 65:537-568.
2. Betsh G. Primitive Near- rings, Math. Z. 1973; 130:351-361.
3. Groenwald NJ. A characterization of semi- prime ideals in near- rings, J Austral. Math. Soc. 1983; 35:194-196.
4. Ramakotaiah D. Radicals for Near- rings, Math. Z 1967; 97:45-56.
5. Bhavanari S, Rao V. The prime radical in near rings, Indian J. Pure Appl. Math. 1984; 15:361-364.
6. Pilz G. Near rings, North Holland Publ. Co., Amsterdam, 1977.
7. Laxton RR. Note on the Radical of a Near- ring, J London Math. Soc. 1972, 612-14.

.