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Upper and lower bounds for norm of weighted type composition operators

Dr. Anshu Sharma

Abstract
Let ψ be holomorphic map of the open unit disk $\mathbb{D}, \varphi$ a holomorphic self-map $\mathbb{D}$ and $H(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. For a non-negative integer n, the weighted type composition operator $D^n_{\varphi,\psi}$ is defined by $D^n_{\varphi,\psi} f = \psi \cdot \left(f^{(n)} \circ \varphi\right), f \in H(\mathbb{D})$.
In this paper, we compute upper and lower bounds for norm of $D^n_{\varphi,\psi}$ from $Q_K(p,q)$ spaces to Bloch-type spaces.

Keywords: composition, holomorphic functions

Introduction
Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}, \partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$ (i.e.$(\mathbb{D}) = 1$), $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Let $\eta_a(z) = (a - z)/(1 - \bar{a}z)$, $a, z \in \mathbb{D}$, that is, the involutive automorphism of $\mathbb{D}$ interchanging points a and 0 . It is well known that $\frac{1-|\eta_a(z)|^2}{1-|z|^2} = \frac{1-|a|^2}{|1-\bar{a}z|^2} = |\eta'_a(z)|$. Also let the Green function in ID with logarithmic singularity at a is given by $g(z,a) = \log \left|\frac{1-\bar{a}z}{a-z}\right| = \log \frac{1}{|\eta_a(z)|}$.
Let ω be a strictly positive continuous function on $\mathbb{D}$. If $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{D}$, we call it a radial weight. A radial weight ω is called typical if it is non-increasing with respect to $|z|$ and $\omega(z) \to 0$ as $|z| \to 1$.
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Key words and phrases composition operator, weighted composition operator, $Q_K(p,q)$ space, $F(p,q,s)$ space, Bloch-type space.

Norm of weighted type operators
For a typical weight ω, the Bloch-type space $\mathcal{B}_\omega$ on $\mathbb{D}$ is the space of $f \in H(\mathbb{D})$ such that $\| f \|_\omega \sup_{z\in\mathbb{D}} \omega(z)|f'(z)| < \infty \mathcal{B}_\omega$ is a Banach space with the norm $\| f \|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z\in\mathbb{D}} \omega(z)|f'(z)|$. When $\omega(r) = (1 - r^2)^\alpha, \alpha > 0$ the induced space $\mathcal{B}_\omega$ becomes α-Bloch space $\mathcal{B}^\alpha$. If $0 < \alpha < 1$, then $\mathcal{B}^\alpha$ consists of all functions $f \in H(\mathbb{D})$ satisfying Lipschitz condition $|f(z) - f(w)| \lesssim |z - w|^{1-\alpha}$ for all $z, w \in \mathbb{D}$ (see [5]).
Let $0 < p < \infty, -2 < q < \infty$ and $K: [0,\infty] \to [0,\infty]$ a non-decreasing continuous function. A function $f \in H(\mathbb{D})$ is in $Q_K(p,q)$ if $M(f) = \left\{\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) dA(z)\right\}^{1/p} < \infty$. Throughout this paper, we assume that

$$\int_0^1 (1 - r^2)^q K(-\log r) r dr < \infty, \tag{1}$$

Since otherwise $Q_K(p,q)$ consists only of constant functions (see [19]). For $1 \le p < \infty, Q_K(p,q)$ is a Banach space with respect to the norm $\| f \|_{Q_K(p,q)} = |f(0)| + M(f)$. If $K(x) = x^s, s \ge 0$, then the spaces $Q_K(p,q)$ reduces to $F(p,q,s)$ which was introduced by R.

Zhao in [20], is known as general family of function spaces. The importance of $F(p,q,s)$ spaces stems from the fact that for appropriate parameter values of p,q and s, $F(p,q,s)$ co-insides with several classical function spaces. For example, $\mathcal{F}(2,1,0)$ is the Hardy space $H^2, \mathcal{F}(p,p+\alpha,0), (\alpha \geq -1)$ is the Bergman space $\mathcal{A}_\alpha^p, \mathcal{F}(p,q,s) = \mathcal{B}^{\frac{2+q}{p}}$ for $s > 1, \mathcal{F}(p,q,s) \subset \mathcal{B}^{\frac{2+q}{p}}$ for $0 < s \leq 1$, $\mathcal{F}(2,0,p) = Q_p$ and $\mathcal{F}(2,0,1) = BMOA$, the space of analytic functions with bounded mean oscillation. If $q+s \leq -1, \mathcal{F}(p,q,s)$ is the space of constant functions.

For a typical weight ω, the weighted spaces of controlled growth $\mathcal{A}_\omega$ consists of $f \in H(\mathbb{D})$ such that $\| f \|_{\mathcal{A}_\omega} = \sup_{z\in\mathbb{D}} \omega(z)|f(z)| < \infty$. We denote by $\mathcal{LB}_\infty^\alpha$ the logarithmic weighted Bloch spaces of holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \left(\log \frac{2}{1-|z|^2}\right)|f'(z)| < \infty$$

Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic map of $\mathbb{D}$. For a non-negative integer n, we define a linear operator $D_{\varphi,\psi}^n$ as follows:

$$D_{\varphi,\psi}^n f = \psi \cdot \left(f^{(n)} \circ \varphi\right),\ f \in H(\mathbb{D}).$$

We call it generalized weighted composition operator, since it include many known operators. For example, if $n=0$ and $\psi \equiv 1$, then we obtain the composition operator C_φ induced by φ, defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. If $\psi = 1$ and $\varphi(z) = z$, then $D_{\varphi,\psi}^n = D^n$, the differentiation operator defined as $D^n f = f^{(n)}$. If $n=0$, then we get the weighted composition operator ψC_φ defined as $\psi C_\varphi f = \psi \cdot (f \circ \varphi)$. If $n=1$ and $\psi(z) = \varphi'(z)$, then $D_{\varphi,\psi}^n$ reduces to DC_φ. When $\psi \equiv 1$, then $D_{\varphi,\psi}^n$ reduces to $C_\varphi D^n$. If we put $\varphi(z) = z$, then $D_{\varphi,\psi}^n = M_\psi D^n$, the product of multiplication and differentiation operator. We provide a unified way of treating these operators from $Q_K(p,q)$ spaces to Bloch-type spaces.

For more about operators of the type $D_{\varphi,\psi}^n$ we refer [1, 19]. In this paper, we characterize boundedness and compactness of $D_{\varphi,\psi}^n$ from $Q_K(p,q)$ spaces to Bloch-type spaces.

Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that $B \lesssim A \lesssim B$, where $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$.

2. Upper and lower bounds for $D_{\varphi,\psi}^n$

In this section, we compute the upper and lower bounds for norm of $D_{\varphi,\psi}^n : Q_K(p,q) \to B_\omega$.

For the purpose we need several lemmas. The first among these can be found in [19].

Lemma 1. Let $0 < p < \infty, -2 < q < \infty, \alpha = (2+q)/p, K: [0,\infty] \to [0,\infty]$ a non-decreasing continuous function such that (1) holds and $\in Q_K(p,q)$. Then

(a) $Q_K(p,q) \subset B^\alpha$,

(b) $Q_K(p,q) = B^\alpha$ if and only if

$$\int_0^1 (1-r^2)^{-2} K(-\log r) r dr < \infty. \tag{2}$$

Moreover $\| f \|_{\mathcal{B}^\alpha} \lesssim \| f \|_{Q_K(p,q)}$.

The growth of functions and their derivatives in α-Bloch spaces B^α and $Q_K(p,q)$ spaces is essential to our study. To this end, the following estimates will be useful (see, for example) [21].

Lemma 2. Let $\alpha > 0$ and $f \in \mathcal{B}^\alpha$. Then

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$$|f(z)| \lesssim \begin{cases} \| f \|_{\mathcal{B}^\alpha} & \text{if } 0 < \alpha < 1 \\ \log \dfrac{2}{1-|z|^2} \| f \|_{\mathcal{B}^\alpha} & \text{if } \alpha = 1 \\ \dfrac{\| f \|_{B\alpha}}{(1-|z|^2)^{\alpha-1}} & \text{if } \alpha > 1 \end{cases}$$

And

$$\left|f^{(n)}(z)\right| \lesssim \frac{\| f \|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}} \text{ if } n \geq 1$$

The next lemma can easily be deduced from Lemma 1 and Lemma 2.

Lemma 3. Let $0 < p < \infty$ and $-2 < q < \infty, K: [0,\infty] \to [0,\infty]$ is a non-decreasing continuous function such that (1) holds and $f \in Q_K(p,q)$.
Then

$$|f(z)| \lesssim \begin{cases} \| f \|_{Q_K(p,q)} & \text{if } 2+q < p \\ \log \frac{2}{1-|z|^2} \| f \|_{Q_K(p,q)} & \text{if } 2+q = p \\ \frac{\| f \|_{Q_K(p,q)}}{(1-|z|^2)^{\frac{2+q}{p}-1}} & \text{if } 2+q > p \end{cases}$$

And

$$\left|f^{(n)}(z)\right| \lesssim \frac{\| f \|_{Q_K(p,q)}}{(1-|z|^2)^{\frac{2+q}{p}+n-1}} \text{ if } n \geq 1.$$

Theorem 1. Let $0 < p < \infty, -2 < q < \infty, K: [0,\infty] \to [0,\infty]$ a nondecreasing continuous function such that (1) holds, w a typical weight, $\psi \in H(\mathbb{D}), n \in \mathbb{N}$, or $n = 0$ and φ be a holomorphic self-map on $\mathbb{D}$).
(1). If $n \in \mathbb{N}$, or $n = 0$ and $2 + q > p$, then $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}_\omega$ is bounded if and only if

(a) $M_1 := \sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{v}+n-1}} < \infty;$

(b) $M_2 := \sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} < \infty.$

Moreover, the following relation holds

$$M_1 + M_2 \lesssim \left\|D^n_{\varphi,\psi}\right\|_{Q_K(p,q)\to B_\omega} \lesssim M_1 + M_2 + \frac{|\psi(0)|}{(1-|\varphi(0)|^2)^{\frac{2+q}{p}+n-1}}. \quad (3)$$

(2). If $2 + q < p$, then $D^0_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}_\omega$ is bounded if and only if $\psi \in \mathcal{B}$ and

(c) $M_3 := \sup_{z\in\mathbb{D}} \frac{\omega(z)|\varphi'(z)\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} < \infty.$

Moreover, the following relation holds

$$\| \psi \|_{\mathcal{B}_\omega} + M_3 \lesssim \left\|D^0_{\varphi,\psi}\right\|_{Q_K(p,q)\to B_\omega} \lesssim \| \psi \|_{\mathcal{B}_\omega} + M_3 + |\psi(0)|. \quad (4)$$

(3). If $2 + q = p$ and equation (2) in Lemma 1 holds, then $D^0_{\varphi,\psi} : Q_K(p,q) \to B_\omega$ is bounded if and only if (c) holds and

(d) $M_4 := \sup_{z\in\mathbb{D}} \frac{\omega(z)|\varphi'(z)\psi(z)|}{1-|\varphi(z)|^2} < \infty;$

(e) $M_5 := \sup_{z\in\mathbb{D}} \omega(z)|\psi'(z)|\log\left(\frac{2}{1-|\varphi(z)|^2}\right) < \infty.$

Moreover, the following relation holds

$$M_4 + M_5 \lesssim \left\|D^0_{\varphi,\psi}\right\|_{Q_K(p,q)\to B_\omega} \lesssim M_4 + M_5 + |\psi(0)|\log\left(\frac{2}{1-|\varphi(0)|^2}\right). \quad (5)$$

Proof. (1). Suppose that $n \in \mathbb{N}$, or $n = 0$ and $2 + q > p$ and conditions (a) and (b) hold. For arbitrary $z \in \mathbb{D}$ and $f \in Q_K(p,q)$, by Lemma 3, we have

$$\omega(z)\left|\left(D^n_{\varphi,\psi}f\right)'(z)\right| \leq \omega(z)|\psi'(z)|\left|f^{(n)}(\varphi(z))\right| + \omega(z)|\varphi'(z)\psi(z)|\left|f^{(n+1)}(\varphi(z))\right|$$

$$\lesssim \left(\frac{\omega(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} + \frac{\omega(z)|\varphi'(z)\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}}\right)||f||_{Q_K(p,q)} \quad (6)$$

$\lesssim (M_1 + M_2)|ff|Q_K(p,q)$

Moreover,

$$|(D^n_{\varphi,\psi}f)(0)| = |\psi(0)||f^{(n)}(\varphi(0))| \lesssim \frac{|\psi(0)||f||_{Q_K(p,q)}}{(1-|\varphi(0)|^2)^{\frac{2+q}{p}+n-1}}$$

Hence $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}_\omega$ is bounded and

$$||D^n_{\varphi,\psi}||_{Q_K(p,q)\to B_\omega} \lesssim M_1 + M_2 + \frac{|\psi(0)|}{(1-|\varphi(0)|^2)^{\frac{2+q}{p}+n-1}}. \tag{7}$$

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Conversely, suppose that $D^n_{\varphi,\psi}: Q_K(p,q) \to B_\omega$ is bounded. Then by taking $f(z) = z^n/n!$ and $f(z) = z^{n+1}/(n+1)!$, and using the fact that $|\varphi(z)| < 1$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi'(z)| \lesssim \|D^n_{\varphi,\psi}\|_{Q_K(p,q)\to B_\omega} \tag{8}$$

And

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)\psi(z)| < \|D^n_{\varphi,\psi}\|_{Q_K(p,q)\to B_\omega}. \tag{9}$$

For

$\lambda \in \mathbb{D}$, let $f_\lambda(z) = \frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(\lambda)}\,z)^{\frac{2+q}{p}}}$.

Then $f_\lambda \in Q_K(p,q)$ for each $\lambda \in \mathbb{D}$, moreover there is a constant $C > 0$ such that $\sup_{z\in\mathbb{D}}\|f_\lambda\|_{Q_K(p,q)} \le C$(see) [8]. For $k \in \mathbb{N}$, we have

$$f_\lambda^{(k)}(z) = \prod_{j=1}^{k}\left(\frac{2+q}{p}+j-1\right)\overline{(\varphi(\lambda))}^k \frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(\lambda)}\,z)^{\frac{2+q}{p}+k}}.$$

Therefore,

$$\begin{aligned}\|D^n_{\varphi,\psi}\|_{Q_K(p,q)\to B_\omega} &\gtrsim ||D^n_{\varphi,\psi}f_\lambda||_{\mathcal{B}\omega} \ge \omega(\lambda)\left|(D^n_{\varphi,\psi}f_\lambda)'(\lambda)\right| \\ &\ge \omega(\lambda)|\psi'(\lambda)f^{(n)}(\varphi(\lambda)) + \varphi'(\lambda)\psi(\lambda)f^{(n+1)}(\varphi(\lambda))| \\ &= \omega(\lambda) \mid \psi'(\lambda)\prod_{j=1}^{n}\left(\frac{2+q}{p}+j-1\right)\frac{\overline{(\varphi(\lambda))}^n(1-|\varphi(\lambda)|^2)}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \\ &+\psi(\lambda)\varphi'(\lambda)\prod_{j=1}^{n+1}\left(\frac{2+q}{p}+j-1\right)\frac{(\varphi(\lambda))^{n+1}(1-|\varphi(\lambda)|^2)}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n+1}} \mid\end{aligned}$$

$$= \mid \prod_{j=1}^{n}\left(\frac{2+q}{p}+j-1\right)(\varphi(\lambda))^n\frac{\omega(\lambda)\psi'(\lambda)}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n-1}} + \prod_{j=1}^{n+1}\left(\frac{2+q}{p}+j-1\right)(\varphi(\lambda))^{n+1}\frac{\omega(\lambda)\psi(\lambda)\varphi'(\lambda)}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \mid \tag{10}$$

for every $\lambda \in \mathbb{D}$, from which it follows that

$$\prod_{j=1}^{n}\left(\frac{2+q}{p}+j-1\right) \mid \overline{\varphi(\lambda)} \mid \quad \frac{\omega(z)|\psi'(\lambda)|}{|1-|\varphi(\lambda)|^2|^{\frac{2+q}{p}+n-1}}$$

$$\lesssim ||D^n_{\varphi,\psi}||_{Q_K(p,q)\to B_\omega} + \prod_{j=1}^{n+1}\left(\frac{2+q}{p}+j-1\right)|\varphi(\lambda)|^{n+1}\frac{\omega(z)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \tag{11}$$

Again, for $\lambda \in \mathbb{D}$, let

$$g_\lambda(z) = \frac{1-|\varphi(z)|^2}{\left(1-\overline{\varphi(\lambda)}z\right)^{\frac{2+q}{p}}} - \frac{2+q}{2+q+np}\frac{(1-|\varphi(z)|^2)^2}{\left(1-\overline{\varphi(\lambda)}z\right)^{\frac{2+q}{p}+1}}$$

Then $g_\lambda \in Q_K(p,q)$ and there is a constant $C > 0$ such that $\sup_{z\in\mathbb{D}} \|g_\lambda\|_{Q_K(p,q)} \leq$ C. Moreover, for $k \in \mathbb{N}$, we have

$$g_\lambda^{(k)}(z) = \prod_{j=1}^{k}\left(\frac{2+q}{p}+j-1\right)\overline{(\varphi(\lambda))}^k \frac{1-|\varphi(z)|^2}{\left(1-\overline{\varphi(\lambda)z}\right)^{\frac{2+q}{p}+k}} - \left(\frac{2+q}{2+q+np}\right)\prod_{j=1}^{k}\left(\frac{2+q}{p}+j\right)\frac{\overline{(\varphi(\lambda))}^k(1-|\varphi(z)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^{\frac{2+q}{p}+1+k}}$$

Thus $g_\lambda^{(n)}(\varphi(\lambda)) = 0$ and

$$g_\lambda^{(n+1)}(\varphi(\lambda)) = -\frac{p}{2+q+np}\prod_{j=1}^{n+1}\left(\frac{2+q}{p}+j-1\right)\frac{\overline{(\varphi(\lambda))}^{n+1}}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \tag{12}$$

Therefore,

$$\|D_{\varphi,\psi}^n\|_{Q_K(p,q)\to B_\omega} \geq \|D_{\varphi,\psi}^n g_\lambda\|_{\mathcal{B}_\omega}$$

$$\geq \frac{p}{2+q+np}\prod_{j=1}^{n+1}\left(\frac{2+q}{p}+j-1\right)\left(\overline{\varphi(\lambda)}\right)^{n+1}\frac{\omega(\lambda)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \tag{13}$$

and so

$$\sup_{\lambda\in\mathbb{D}}\frac{\omega(\lambda)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}}|\varphi(\lambda)|^{n+1} \lesssim \|D_{\varphi,\psi}^n\|_{Q_K(p,q)\to B_\omega} \tag{14}$$

Thus for fixed $\delta, 0 < \delta < 1$, by (14), we have

$$\sup_{|\varphi(\lambda)|>\delta}\frac{\omega(\lambda)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \lesssim \|D_{\varphi,\psi}^n\|_{Q_K(p,q)+B_\omega} \tag{15}$$

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For $\lambda \in \mathbb{D}$ such that $|\varphi(\lambda)| \leq \delta$, we have

$$\sup_{|\varphi(\lambda)|\leq\delta}\frac{\omega(\lambda)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \leq \frac{C}{(1-\delta^2)^{\frac{2+q}{p}+n}}(1-|\lambda|^2)|\psi(\lambda)\varphi'(\lambda)|. \tag{16}$$

Hence from (8) and (16), we have

$$\sup_{|\varphi(\lambda)|\leq\delta}\frac{\omega(\lambda)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \lesssim \|D_{\varphi,\psi}^n\|_{Q_K(p,q)\to B_\omega} \tag{17}$$

Thus from (15) and (17), we have

$$M_2 := \sup_{\lambda\in\mathbb{D}}\frac{\omega(\lambda)|\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} \lesssim \|D_{\varphi,\psi}^n\|_{Q_K(p,q)\to B_\omega} \tag{18}$$

From (10) and (18) we can obtain

$$M_1 := \sup_{\lambda\in\mathbb{D}}\frac{\omega(\lambda)|\psi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n-1}} \lesssim ||D_{\varphi,\psi}^n||_{Q_K(p,q)\to B_\omega} \tag{19}$$

Combining (18) and (19), we have

$$M_1 + M_2 \lesssim \|D_{\varphi,\psi}^n\|_{Q_K(p,q)+B_\omega} \tag{20}$$

Hence from (7) and (20), we have

$$M_1 + M_2 \lesssim \left\| D_{\varphi,\psi}^{n} \right\|_{Q_K(p,q)\to B_\omega} \lesssim M_1 + M_2 + \frac{|h(0)|}{(1-|\varphi(0)|^2)^{\frac{2+q}{p}+n-1}}.$$

Next we will prove (2). Suppose that $n = 0, 2 + q < p$, condition (c) holds and $\psi \in \mathcal{B}_\omega$. Assume that $f \in Q_K(p,q)$. Then by Lemma 3, we have $\sup_{z\in\mathbb{D}}|f(z)| \le C||f|_{Q_K(p,q)}$. It follows that $\omega(z) \mid \left(D_{\varphi,\psi}^{n} f\right)'(z) \mid$

$$\le \omega(z)|\psi'(z)||f(\varphi(z))| + \omega(z)|\varphi'(z)\psi(z)||f'(\varphi(z))|$$

$$\lesssim \left(\omega(z)|\psi'(z)| + \frac{\omega(z)|\varphi'(z)\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} \right) \| f \|_{Q_K(p,q)}$$

$$\lesssim \left(\| \psi \|_{\mathcal{B}_\omega} + M_3 \right) \| f \|_{Q_K(p,q)} \tag{21}$$

Moreover,

$$\left| \left(D_{\varphi,\psi}^{0} f\right)'(0) \right| = |\psi(0)||f(\varphi(0))| \lesssim |\psi(0)| \| f \|_{Q_K(p,q)}.$$

Hence $D_{\varphi,\psi}^{0}: Q_K(p,q) \to \mathcal{B}_\omega$ is bounded and

$$\left\| D_{\varphi,\psi}^{0} \right\|_{Q_K(p,q)\to B_\omega} \lesssim \| \psi \|_{\mathcal{B}_\omega} + M_3 + |h(0)|. \tag{22}$$

That $\psi \in \mathcal{B}$ and (c) holds can be proved proceeding as the proof of (1).

Moreover, we have $\| \psi \|_{\mathcal{B}\omega} + M_3 \lesssim \left\| D_{\varphi,\psi}^{0} \right\|_{Q_K(p,q)\to B_\omega}$ and so $\| \psi \|_{\mathcal{B}_\omega} + M_3 \lesssim \left\| D_{\varphi,\psi}^{0} \right\|_{Q_K(p,q)+B_\omega} \lesssim \| \psi \|_{\mathcal{B}_\omega} + M_3 + |\psi(0)|$.
To prove (3), let $n = 0, 2 + q = p$. Assume that equation (2) of Lemma 1 and conditions (c) and (d) hold and $f \in Q_K(p,q)$. Then by Lemma 3, we have $\omega(z)\left| \left(D_{\varphi,\psi}^{0} f\right)'(z) \right|$

$$\le \omega(z)|\psi'(z)||f(\varphi(z))| + \omega(z)|\varphi'(z)\psi(z)||f'(\varphi(z))|$$

$$\lesssim \left(\omega(z)|\psi'(z)|\log\left(\frac{2}{1-|\varphi(z)|^2}\right) + \frac{\omega(z)|\varphi'(z)\psi(z)|}{1-|\varphi(z)|^2} \right) ||f||_{Q_K(p,q)}$$

$$\lesssim (M_4 + M_5)||f||_{Q_K(p,q)} \tag{23}$$

Moreover,

$$\left| \left(D_{\varphi,\psi}^{0} f\right)(0) \right| = |\psi(0)||f(\varphi(0))| \lesssim |\psi(0)|\log\left(\frac{2}{1-|\varphi(0)|^2}\right) ||f||_{Q_K(p,q)}.$$

Hence $D_{\varphi,\psi}^{0}: Q_K(p,q) \to \mathcal{B}_\omega$ is bounded and

$$\left\| D_{\varphi,\psi}^{0} \right\|_{Q_K(p,q)\to B_\omega} \lesssim M_4 + M_5 + |\psi(0)|\log\left(\frac{2}{1-|\varphi(0)|^2}\right) \tag{24}$$

Conversely, suppose that $D_{\phi,\psi}^{0}: Q_K(p,q) \to \mathcal{B}_\omega$ is bounded. For $\lambda \in \mathbb{D}$, let

$$f_\lambda(z) = 2\log\left(\frac{2}{1-\overline{\varphi(\lambda)z}}\right) - \left(\log\left(\frac{2}{1-|\varphi(z)|^2}\right)\right)^{-1} \left(\log\left(\frac{2}{\overline{\varphi(\lambda)z}}\right)\right)^2$$

Then

$$f_\lambda'(z) = \frac{2\overline{\varphi(\lambda)}}{1-\overline{\varphi(\lambda)}z} - 2\log\left(\frac{2}{1-\overline{\varphi(\lambda)}z}\right)\frac{\overline{\varphi(\lambda)}}{1-\overline{\varphi(\lambda)}z}\left(\log\left(\frac{2}{1-|\varphi(z)|^2}\right)\right)^{-1}$$

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Easy calculation yields that there exists $C > 0$ such that $\sup_{z\in\mathbb{D}} \|f_\lambda\|_{\mathcal{B}} \leq$ C. Since equation (2) of Lemma 1 holds, we have $f_\lambda \in Q_K(p,q)$. and $\sup_{z\in\mathbb{D}} \|f_\lambda\|_{Q_K(p,q)} \leq C$. Also

$$f_\lambda'(\varphi(\lambda)) = 0 \text{ and } f_\lambda\left(\varphi(\lambda)\right) = \log\left(\frac{2}{1-|\varphi(z)|^2}\right)$$

Thus we can easily show that $M_5 \lesssim \|D^0_{\varphi,\psi}\|_{Q_K(p,q)+B_\omega}$. Moreover, we can prove that $M_4 \lesssim \|D^0_{\varphi,\psi}\|_{Q_K(p,q)+B_\omega}$ and so we have $M_4 + M_5 \lesssim \|D^0_{\varphi,\psi}\|_{Q_K(p,q)\to B_\omega}$. This completes the proof.

Corollary 1. Let $0 < p < \infty, -2 < q < \infty, K: [0,\infty] \to [0,\infty]$ a nondecreasing continuous function such that (1) holds, $\varphi(z) = z, \psi \in H(\mathbb{D})$ and $n \in \mathbb{N}$, or $n = 0$. Then $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}^\alpha$ is bounded if and only if $\psi \in \mathfrak{M}$, where

$$\mathfrak{M} = \begin{cases} \mathcal{B}^{\alpha+1-n-(2+q)/p} & \text{if } n + (2+q)/p < \alpha, n \in \mathbb{N}, \\ H^\infty & \text{if } n + (2+q)/p = \alpha, n \in \mathbb{N} \\ \{0\}, & \text{if } n + (2+q)/p > \alpha, n \in \mathbb{N}, \\ \mathcal{B}^{\alpha+1-(2+q)/p} & \text{if } 1 < (2+q)/p < \alpha, n = 0, \\ H^\infty, & \text{if } 1 < (2+q)/p = \alpha, n = 0, \\ \{0\}, & \text{if } (2+q)/p > \max\{\alpha, 1\}, n = 0, \\ \mathcal{LB}^\alpha & \text{if } (2+q)/p = 1 < \alpha, n = 0 \text{ and condition} \\ & \text{(2) of Lemma 1 holds}, \\ \mathcal{LB} \cap \mathcal{H}^\infty, & \text{if } (2+q)/p = 1 = \alpha, n = 0 \text{ and condition} \\ & \text{(2) of Lemma 1 holds}, \\ \{0\} & \text{if } (2+q)/p = 1 > \alpha, n = 0 \text{ and condition} \\ & \text{(2) of Lemma 1 holds}, \end{cases}$$

Proof. First suppose that $n \in \mathbb{N}$. By taking $\omega(z) = (1-|z|^2)^\alpha$ and $\varphi(z) = z$ in (1) of Theorem 1, we have $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1-n-(2+q)/p} |\psi'(z)| < \infty \tag{25}$$

And

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-n-(2+q)/p} |\psi(z)| < \infty \tag{26}$$

Suppose that $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}^\alpha$ is bounded. If $n + (2+q)/p > \alpha$, then (26) implies that there exists a constant $C > 0$ such that $|\psi(z)| \leq C(1-|z|^2)^{n+(2+q)/p-\alpha}$, so by maximum modulus principle $\psi \equiv 0$. If $n + (2+q)/p = \alpha$, then from (25) and (26) we have $\psi \in \mathcal{B}$ and $\psi \in H^\infty$ respectively. Since $H^\infty \subset \mathcal{B}$, we obtain that $\psi \in H^\infty$. If $n + (2+q)/p < \alpha$, then from (25) and (26) we have $\psi \in \mathcal{B}^{\alpha+1-n-(2+q)/p}$ and $\psi \in \mathcal{A}^{\alpha-n-(2+q)/p}$ respectively. Since $\mathcal{B}^{\alpha+1-n-(2+q)/p} = \mathcal{A}^{\alpha-n-(2+q)/p}$, we have $\psi \in \mathcal{B}^{\alpha+1-n-(2+q)/p}$. Conversely suppose that $\psi \in \mathfrak{M}$. If $n + (2+q)/p > \alpha$, then it is obvious that (25) and (26) hold and so $D^n_{\varphi,\psi}: Q_K(p,q) \to B^\alpha$ is bounded. If $n + (2+q)/p = \alpha$, then $\psi \in H^\infty$. Now we can sasily see that $\psi \in H^\infty$ implies (25) and (26) and so $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}^\alpha$ is bounded. If $n + (2+q)/p < \alpha$, then we have $\psi \in \mathcal{B}^{\alpha+1-n-(2+q)/p} = \mathcal{A}^{\alpha-n-(2+q)/p}$ and so $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}^\alpha$ is bounded. Again, if $n = 0$ and $2 + q > p$, then proceeding as above we get the desired result. We omit the details. If $n = 0$ and $2 + q < p$, then by taking $\omega(z) = (1-|z|^2)^\alpha$ and $\varphi(z) = z$ in (2) of Theorem 1, we have $D^0_{\varphi,\psi}: Q_K(p,q) \to B^\alpha$ is bounded if and only $\psi \in \mathcal{B}^\alpha$ and $\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-(2+q)/p} |\psi(z)| < \infty$. From these conditions we easily get desired result. We omit the details. If $n = 0, 2 + q = p$ and condition (2) of Lemma 1 holds, then by taking $\omega(z) = (1-|z|^2)^\alpha$ and $\varphi(z) = z$ in (2) of Theorem 1, we have $D^0_{\varphi,\psi}: Q_K(p,q) \to B^\alpha$ is bounded if and only $\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha-1} |\psi(z)| < \infty$ and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \log\left(\frac{2}{1-|z|^2}\right) |\psi'(z)| < \infty$$

If $\alpha > 1$, then the second condition implies first. If $\alpha = 1$, then the first condition implies $\psi \in H^\infty$. If $\alpha < 1$, then above conditions condition imply $\psi \equiv 0$. From these conditions we easily get desired result. We omit the detail

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