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The exponentıal generating functıons of jacobsthal and jacobsthal lucas ıdentıtıes

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Abstract
In this study, we use exponential generating functions for Jacobsthal and Jacobsthal Lucas sequences to derive new sum formulas for Jacobsthal and Jacobsthal Lucas sequences.

Keywords: Jacobsthal numbers, exponential generating functions
AMS Classifications: 11B39, 11B83, 15A24, 15B36

Introduction

Generating functions are one of the most useful inventions in discrete mathematics. Roughly speaking, generating functions transform problems about sequences into problems about functions. The ordinary generating function for the infinite sequence $\{g_0, g_1, g_2 ...\}$ is the power series $G(x)=g_0+g_1 x+g_2 x^2+$....The generating functions are also very important for integer sequence with no doubt. In [5], Carlıtz gave some properties.of the generating functions. In [1, 4], the authors gave some properties about generating functions and ordinary generating functions for identities about Fibonacci and Lucas numbers. In [2], Gould has studied generalized generating functions as products of powers of Fibonacci numbers. In [3], Harold, found Fibonacci and Lucas identities and generating functions, in his master thesis. In [6], Church established some exponential generating functions established Fibonacci and Lucas identities. In [8], Kolodner investigated a generating function associated with generalized Fibonacci numbers. In this paper we investigate some properties of Jacobsthal and Jacobsthal Lucas sequences by using exponential generating functions. Integer sequences have been an important research subject for many of papers. We can meet integer sequences many of area archtucture, nature, art, body of human. The oldest known integer sequences occur Fibonacci and Lucas numbers which are defined as $f_n = f_{n-1} + f_{n-2}$; $f_0 = 0,\ f_1 = 1$ and $l_n = l_{n-1} + l_{n-2};\ l_0 = 2,\ l_1 = 1$ for $n \ge 2$; By changing the recursion formula for Fibonacci and Lucas numbers, Jacobsthal and Jacobsthal Lucas have been first defined by German mathematician Ernst Jacobsthal. Horadam, in [3] exhibited some identities of Jacobsthal and Jacobsthal Lucas numbers

Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$; $j_0 = 0,\ j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$; $c_0 = 2,\ c_1 = 1$ for $n \ge 2$ respectively in [10]. The characteristic equation of recurrence relations $x^2-x-2=0$, with roots $\alpha=2$, $\beta=-1$.

Binet's formulas allow us to express the Jacobsthal numbers and Jacobsthal Lucas numbers in function of the roots α,β are defined by

$$j_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{2^n - (-1)^n}{3},\ c_n = \alpha^n + \beta^n = 2^n + (-1)^n \quad (1)$$

The ordinary generating functions of these sequences are

$$\sum_{k=0}^{\infty} j_k x^k = \frac{x}{1-x-2x^2}$$
$$\sum_{k=0}^{\infty} c_k x^k = \frac{2-x}{1-x-2x^2}$$

By using the expansion of Maclaurin series of the exponential function we have

$$e^{\alpha t} = 1 + \frac{\alpha t}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} + \ldots + \frac{(\alpha t)^n}{n!} + \ldots$$

And

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = (1-1) + \frac{(\alpha - \beta)t}{1!} + \frac{(\alpha^2 - \beta^2)t^2}{2!} + \frac{(\alpha^3 - \beta^3)t^3}{3!} + \frac{(\alpha^4 - \beta^4)t^4}{4!} + \ldots$$

So that, we obtain the exponential generating function as

$$\frac{e^{2t} - e^{-t}}{3} = \sum_{n=0}^{\infty} j_n \frac{t^n}{n!} \tag{2}$$

$$e^{2t} + e^{-t} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \tag{3}$$

In [6], the well known double sum property is given as

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(k,n-k).$$

In [9], the authors found the following properties about Jacobsthal and Jacobsthal Lucas numbers

$$\sum_{k=0}^{n} \binom{n}{k} j_k = 3^{n-1},$$
$$\sum_{k=0}^{n} \binom{n}{k} (-2)^{n-1} j_k = 3^{n-1},$$
$$\sum_{k=0}^{n} \binom{n}{k} j_k c_{n-k} = 2^n j_n.$$

Lemma 1 Let $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ and $B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$ then

$$A(t)B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!} \tag{4}$$

And

$$A(t)B(-t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!}$$

Exponentıal Generatıng Functıons For Jacobsthal Identıtıes

The characteristic equation of recurrence relations of Jacobsthal and Jacobsthal Lucas numbers are $x^2-x-2=0$,with roots $\alpha=2, \beta=-1$. Useful algebraic properties of α, β are

$$\alpha\beta = -2,\ \alpha+\beta=1,\ \alpha-\beta=3 \tag{5}$$

$$\alpha^n = \alpha j_n + 2 j_{n-1},\quad \alpha^2=\alpha+2$$

Theorem 2 For n positive integers, the following identities are obtained as

$$\sum_{k=0}^{n}\binom{n}{k} j_k 2^{n-k} = j_{2n}$$

$$\sum_{k=0}^{n}\binom{n}{k} c_k 2^{n-k} = c_{2n}$$

Proof: If we choose $A(t)=\dfrac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}=\sum_{n=0}^{\infty} j_n \dfrac{t^n}{n!}$ and $B(t)=e^{2t}=\sum_{n=0}^{\infty}\dfrac{(2t)^n}{n!}$ in that form, then from (5)

$$A(t)B(t)=\left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)\left(e^{2t}\right)=\left(\frac{e^{(\alpha+2)t}-e^{(\beta+2)t}}{\alpha-\beta}\right)=\left(\frac{e^{\alpha^2 t}-e^{\beta^2 t}}{\alpha-\beta}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(e^{\alpha^2 t})^n-(e^{\beta^2 t})^n}{\alpha-\beta}\frac{t^n}{n!}=\sum_{n=0}^{\infty} j_{2n}\frac{t^n}{n!}=\sum_{n=0}^{\infty} j_{2n}\frac{t^n}{n!}$$

By using 4, we have

$$\text{A(t)B(t)} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^k-\beta^k}{(\alpha-\beta)}2^{n-k}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} j_k 2^{n-k}\right)\frac{t^n}{n!}$$

By the equality of the above series, the result is obtained. Similarly let $A(t)=e^{\alpha t}+e^{\beta t}$ and $B(t)=e^{2t}$

$$A(t)B(t)=e^{(\alpha+2)t}+e^{(\beta+2)t}$$

$$=\sum_{n=0}^{\infty}\frac{(e^{\alpha^2 t})^n+(e^{\beta^2 t})^n}{n!}\sum_{n=0}^{\infty} c_{2n}\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(\alpha^k+\beta^k)2^{n-k}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} c_k 2^{n-k}\right)\frac{t^n}{n!}$$

Theorem 3 For *n* positive integers,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\, j_k = (-1)^{n+1} j_n$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\, c_k = (-1)^{n} c_n$$

Proof: Assume that $A(t)=\dfrac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$ and $B(t)=e^{-t}$, then the properties of α, β

$$A(t)B(t)=\left(\frac{e^{(\alpha-1)t}-e^{(\beta-1)t}}{\alpha-\beta}\right)=\left(\frac{e^{-\beta t}-e^{-\alpha t}}{\alpha-\beta}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-\beta t)^n-(-\alpha t)^n}{(\alpha-\beta)n!}=\sum_{n=0}^{\infty}(-1)^{n+1} j_n\frac{t^n}{n!}$$

From the multiplication of series, we have

$$A(t)B(t)=\sum_{n=0}^{\infty} j_n\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^k-\beta^k}{(\alpha-\beta)}(-1)^{n-k}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\, j_k\right)\frac{t^n}{n!}$$

By the equality of the above series, the result is obtained. The other equality can be proved by the same method.
Teorem 4 For *n* positive integers,

$$\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k} j_{2k} = j_n$$

$$\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k} c_{2k} = c_n$$

Proof: Let $A(t)=\dfrac{e^{\alpha^2 t}-e^{\beta^2 t}}{\alpha-\beta}$ ve $B(t)=e^{-2t}$, then the properties of α, β

$$A(t)B(t)=\left(\frac{e^{(\alpha^2-2)t}-e^{(\beta^2-2)t}}{\alpha-\beta}\right)=\left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right) =\sum_{n=0}^{\infty}J_n\frac{t^n}{n!}$$

By using 4, we have

$$A(t)B(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^{2k}-\beta^{2k}}{(\alpha-\beta)}(-2)^{n-k}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-k}\, j_{2k}\right)\frac{t^n}{n!}$$

By the equality of the series, we easily see the desired result.

Theorem 5

$$\sum_{k=0}^{n}\binom{n}{k} j_k c_{n-k} = 2^n j_n$$

Proof: Assume that $A(t)=\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$ and $B(t)=e^{\alpha t}+e^{\beta t}$ then

$$A(t)B(t)=\left(\frac{e^{2\alpha t}-e^{2\beta t}}{\alpha-\beta}\right)=\sum_{n=0}^{\infty}2^n j_n \frac{t^n}{n!}$$

$$A(t)B(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\binom{n}{k}\frac{\alpha^k-\beta^k}{(\alpha-\beta)}(\alpha^{n-k}+\beta^{n-k})\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} j_k c_{n-k}\right)\frac{t^n}{n!}$$

Theorem 6

$$\sum_{k=0}^{n}\binom{n}{k} j_k j_{n-k} = \frac{1}{9}(2^n c_n - 2)$$

Proof: Assume that $A(t)=B(t)=\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$

$$A(t)B(t)=\left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)^2=\frac{1}{9}(e^{2\alpha t}+e^{2\beta t}-2e^{t})$$

$$=\sum_{n=0}^{\infty}(2^n c_n - 2)\frac{t^n}{n!}$$

By using 4, we have

$$A(t)B(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\binom{n}{k}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} j_k j_{n-k}\right)\frac{t^n}{n!}$$

Theorem 7

$$\sum_{k=0}^{n}\binom{n}{k} c_k c_{n-k} = 2^n c_n + 2$$

Proof: Assume that $A(t)=B(t)=e^{\alpha t}+e^{\beta t}$,

$$A(t)B(t)=\left(e^{\alpha t}+e^{\beta t}\right)^2=e^{2\alpha t}+e^{2\beta t}+2e^{t}$$

$$= \sum_{n=0}^{\infty} (2^n c_n + 2) \frac{t^n}{n!}$$

By using 4, we have

$$A(t)B(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\binom{n}{k} (\alpha^k + \beta^k)(\alpha^{n-k} + \beta^{n-k}) \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} \right) \frac{t^n}{n!}$$

Theorem 8 Let *m*,*n* any positive integers

$$\sum_{k=0}^{n} \binom{n}{k} j_{mk} c_{mn-nk} = 2^n j_{mn}$$

Proof: If we choose $A(t) = \frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}$ and $B(t) = e^{\alpha^m t} + e^{\beta^m t}$ in that form, we obtain

$$A(t)B(t) = \frac{e^{2\alpha^m t} - e^{2\beta^m t}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty} 2^n j_{mn} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} j_{mk} c_{mn-nk} \right) \frac{t^n}{n!}$$

Theorem 9 Let *m*,*n* any positive integers

$$\sum_{k=0}^{n} \binom{n}{k} j_{mk} j_{mn-nk} = \frac{1}{9} (2^n c_{mn} - 2c_m^n)$$

Proof: If we choose $A(t) = B(t) = \frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}$ as like and use (4), we have

$$A(t)B(t) = \frac{1}{(\alpha - \beta)^2} \sum_{n=0}^{\infty} \frac{(\alpha^{mn} - \beta^{mn})t^n}{n!} \sum_{n=0}^{\infty} \frac{(\alpha^{mn} - \beta^{mn})t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} j_{mk} j_{mn-mk} \right) \frac{t^n}{n!}$$

$$A(t)B(t) = \left(\frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta} \right)^2 = \frac{1}{9} (e^{2\alpha^m t} + e^{2\beta^m t} - 2e^{(\alpha^m + \beta^m)t})$$

$$= \frac{1}{9} \sum_{n=0}^{\infty} (2^n c_{mn} - 2c_m^n) \frac{t^n}{n!}$$

Theorem 10 Let *m*,*n* any positive integers

$$\sum_{k=0}^{n}\binom{n}{k} c_{mk}\, j_{m(n-k)} = 2^n j_{mn}$$

Proof: If we choose $A(t)=e^{\alpha^m t}+e^{\beta^m t}$ and $B(t)=\dfrac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta}$ as like and use (4), we have

$$A(t)B(t)=\sum_{n=0}^{\infty}\frac{(\alpha^{mn}+\beta^{mn})t^n}{n!}\left(\frac{1}{(\alpha-\beta)^2}\sum_{n=0}^{\infty}\frac{(\alpha^{mn}-\beta^{mn})t^n}{n!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} c_{mk}\, j_{m(n-k)}\right)\frac{t^n}{n!}$$

$$A(t)B(t)=\frac{e^{2\alpha^m t}-e^{2\beta^m t}}{\alpha-\beta}=\sum_{n=0}^{\infty}\frac{2^n(\alpha^{mn}-\beta^{mn})t^n}{(\alpha-\beta)n!}=\sum_{n=0}^{\infty}2^n j_{mn}\frac{t^n}{n!}$$

Theorem 11 Let *m*,*n* any positive integers

$$\sum_{k=0}^{n}\binom{n}{k} c_{mk}\, c_{mn-mk} = 2^n c_{mn} - 2c_m^n$$

Proof: Let $A(t)=B(t)=e^{\alpha^m t}+e^{\beta^m t}$, then by the product of the series we obtain

$$A(t)B(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} c_{mk}\, c_{mn-mk}\right)\frac{t^n}{n!}$$

$$A(t)B(t)=(e^{\alpha^m t}+e^{\beta^m t})^2=e^{2\alpha^m t}+e^{2\beta^m t}-2e^{(\alpha^m+\beta^m)t}=\sum_{n=0}^{\infty}(2^n c_{mn}-2c_m^n)\frac{t^n}{n!}$$

Theorem 12 Let *n*,*r* any positive integers then

$$\sum_{k=0}^{n}\binom{n}{k} 2^{n-k} j_{k+r} = j_{2n+r}$$

Proof: If we choose $A(t)=\dfrac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$ then the rth derivative with respect to *t* is olarak seçilir $D_t^r A(t)=\sum_{k=0}^{n}\binom{n}{k} a_{n+r}\dfrac{t^n}{n!}$.

So that and for $B(t)=e^{2t}$, we have

$$D_t^r A(t)B(t)=e^{2t}D_t^r\left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)=\frac{\alpha^r e^{(\alpha+2)t}-\beta^r e^{(\beta+2)t}}{\alpha-\beta}$$

$$=\frac{\alpha^r e^{\alpha^2 t}-\beta^r e^{\beta^2 t}}{\alpha-\beta}=\sum_{n=0}^{n} j_{2n+r}\frac{t^n}{n!}$$

$$D_t^r A(t)B(t)=\sum_{n=0}^{n}a_{n+r}\frac{t^n}{n!}\sum_{n=0}^{n}\frac{2^n t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} j_{k+r}\, 2^{n-k}\right)\frac{t^n}{n!}$$

Theorem 13 Let *m*, *n*, *r* any positive integers then

$$\sum_{k=0}^{n}\binom{n}{k} j_{4mk+r}\, 2^{2m(n-k)} = c_{2m}^{n}\, j_{2m+4mr}$$

Proof: If we choose $A(t)=\dfrac{e^{\alpha^{4m}t}-e^{\beta^{4m}t}}{\alpha-\beta}$ and $B(t)=e^{2^{2m}t}$, we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} j_{4mk+r}\, 2^{n-k}\right)\frac{t^n}{n!} = D_t^r A(t)B(t)$$

$$=\frac{1}{\alpha-\beta}\left[\left(\alpha^{4m}\right)^r e^{\left(\alpha^{4m}t\right)} - \left(\beta^{4m}\right)^r e^{\left(\beta^{4m}t\right)}\right]e^{(2^{2m})t}$$

$$=\frac{\alpha^{4mr}}{\alpha-\beta}e^{\left(\alpha^{4m}+2^{2m}\right)t} - \frac{\beta^{4mr}}{\alpha-\beta}e^{\left(\beta^{4m}+2^{2m}\right)t}$$

$$=\frac{\alpha^{4mr}}{\alpha-\beta}e^{\left(\alpha^{4m}+(-\alpha\beta)^{2m}\right)t} - \frac{\beta^{4mr}}{\alpha-\beta}e^{\left(\beta^{4m}+(-\alpha\beta)^{2m}\right)t}$$

$$=\frac{\alpha^{4mr}}{\alpha-\beta}e^{\alpha^{2m}\left(\alpha^{2m}+\beta^{2m}\right)t} - \frac{\beta^{4mr}}{\alpha-\beta}e^{\beta^{2m}\left(\alpha^{2m}+\beta^{2m}\right)t}$$

$$=\frac{1}{\alpha-\beta}\left[\alpha^{4mr}e^{\alpha^{2m}C_{2mt}} - \beta^{4mr}e^{\beta^{2m}C_{2mt}}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\alpha^{2mn+4mr}c_{2m}^{n}}{\alpha-\beta}\frac{t^n}{n!} - \sum_{n=0}^{\infty}\frac{\beta^{2mn+4mr}c_{2m}^{n}}{\alpha-\beta}\frac{t^n}{n!} = \sum_{n=0}^{\infty}c_{2m}^{n}J_{2mn+4mr}\frac{t^n}{n!}$$

$$\left[D_t^r A(t)\right]B(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} j_{4mk+r}\, 2^{2m(n-k)}\right)\frac{t^n}{n!}$$

From the equality of the results, the proof is completed.

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