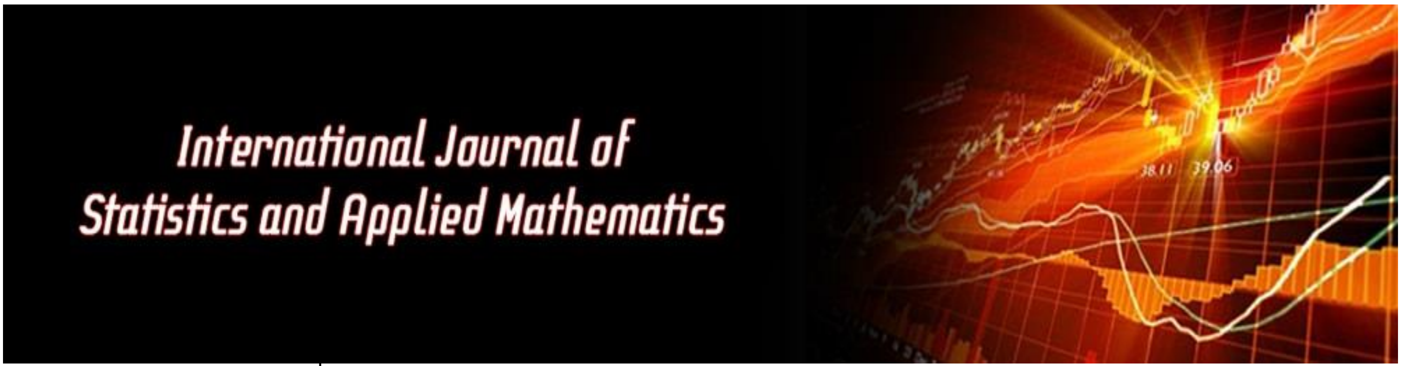

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Integral in topological spaces

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Abstract
Let X, Y be Banach spaces (or either topological vector spaces) and let us consider the function space C (S, X) of all continuous functions f: S → X, from the compact (locally compact) space S into X, equipped with some appropriate topology. Put C (S, X) = C (S) if X = R. In this work we will mainly be concerned with the problem of representing linear bounded operators T: C (S, X) → Y in an integral form: f ∈ C (S, X), Tf =R S f dμ, for some integration process with respect to a measure μ on the Borel σ−field BS of S. The prototype of such representation is the theorem of F. Riesz according to which every continuous functional T: C (S) → R has the Lebesgue integral form Tf =R S f dμ. This paper is intended to present various extensions of this theorem to the Banach spaces setting alluded to above, and to the context of locally convex spaces.

Keywords: Lebesgue integral Riemann integral, topological space

Introduction

A definition of an integral by Majorants and Minorants was first given by Perron in [8] for functions of one real variable. Perron's definition was generalized by Bauer in [1] and later on by Maf ik in [7], for the case of functions of several real variables whose domains of definition are compact intervals. This generalization, however, was not sufficient, because no adequate substitution theorem holds: even a very simple transformation, e.g., a rotation, of a compact interval is not necessarily a compact interval again. A generalization which admits a relatively suitable substitution theorem was given by the author in [9] and [lO]. Although the integral there is defined in a locally compact first countable Hausdorff space, the main emphases are on applications to Euclidean spaces; also only functions with compact domains of definition are integrated. In this paper we shall define a Perron-like integral in an arbitrary topological space and without any restrictions on the domains of integrable functions. Such generality, of course, causes some changes in basic definitions. Because of omission of the first axiom of count-ability, the derivate has to be defined by convergence of nets rather than convergence of sequences. Also the possibility of noncompact domains of integration requires a different definition of the majorant. Throughout P is a topological space and P~ = PW (oo) is a one-point compactification of P. If A QP~, A" and A ~ denote the closure of A in P and P~, respectively. For xÇEP~, T* is a local base at x in P' (see [6, p. 50]). Let a be a nonempty system of subsets of P such that for every A, BE<r, AnB&a and A-B^U?^ d where G, • • •, Cn are dis-joint sets from a.2 We shall assume that rxO for every #£P and that for each TJÇzV* there are disjoint sets Z7 it 00t..., UPtO0 from <r such that î/nP = Uf«1 î7»too where the integer p^l is independent of U.
The algebra generated by a is denoted by <r*; clearly PÇ£a*. If 8 is a collection of subsets of P and AQP", we let h A = \B £0: BC.A}. A system SO is said to be Semihereditary if and only if <JoC\b^0 for every finite disjoint collection (7oO whose union belongs to 5. A system ôO is said to be stable if and only if 0(£8 and for every ^4 G8 and every #£P" there is a f/GT* such that 8A-I/T^0. By a function we shall mean an extended real-valued function. A function F defined on 8Co*" is said to be superadditive or additive whenever F (U?.t A^ E"-i ^W*) or W.M,)-E?-i ^M<). re-Spectively, for every disjoint collection i4i, • • •, an from S for which UîLj ^4»G8 and X^-i^C^») ^as meaning. Semihereditary and stable systems and their connection with superadditive functions were investigated in [11, 12], and [lS]. From now on we shall assume that there is given a nonnegative additive function G defined on a* and such

that G(£/)< + °° for every £/GU \TX: XÇ.P}- Intuitively, G is a finitely additive locally finite measure in P. With every point xÇEP~ we associate a certain family KX of nets {Bu* £/£I\ C} O where T is a cofinal subset of Yx (see [6, Chapter 2]). The collection K= {KX: X^P'} is called a convergence (or some-times a derivation basis—see [4, 1.1]). For x£E.P~ and 8 CO", ** (8) = {{£I/} GK*: {5*} C*}. Throughout we shall assume that the convergence K satisfies the following conditions: 3 5Ci. For every xG-P, {Ut UÇzTx, C} G** and for every integer i, 1 £i£p, {f/.-.co, f/GToo, C} GK,, 3C2. If sG-P" and {BUt U£T, C} G**, then for every FGI1* there is a f/rGr such that BVC V for all UET for which [7C J7F. 5C3. If x£zP~, {-S^jt^er G**, and T' is a cofinal subset of T, then also {Bu} ueT'ÇzKx-3C4. If tfG-P", {Bu} G**, and ^Go", then also {BUC\A} GK*. 5C6. If 8C<r is a nonempty semihereditary, stable system, then the set \X (ELP~: Kx {h) ^0\ is uncountable. If for every x(EP~, ** consists of all nets {Bv} which satisfy con-dition JC2, then the convergence K is called the natural convergence and is denoted by K°. It follows from [12, 2.4] that the natural conver-gence already satisfies all conditions JC1-JC5. Therefore, conditions

The abstract notion of a convergence is the basis for the whole theory. Proposi-tion 4 indicates that we should use a convergence K with KX as small as possible. On the other hand, it is plain thai if the KX are too small, we shall not obtain any integral at all. Conditions 3&-3C* form a set of minimal requirements which a convergence K must satisfy in order to give a meaningful integral. C1-3C5 are not contradictory and it was shown in [17], that they are also independent. Other useful examples of convergences which satisfy conditions JCi~5C6 can be found in [9], [IO], and [17]. Let xÇzP~, A C-P» and let F be a function on a A» We call the num-ber tF(x, A) =inf {lim inf F(Bu): {Bu} E.KX{(TA) } the lower limit of F at x relative to A and the number *F(x, A)=#(F/G)(x, A)A the lower derivate of F at x relative to A. Let A GO" and let *f* be a function on A". A superadditive function M on (TA is said to be a majorant of *f* on A if and only if there is a countable set ZMCZA" such that # (—G) (x, -4)^0 for all XÇ:ZM, iM(x, A)^Q for all xGZAAJ(oo), and -00 ^*M(x, ^) è/(x) for all XÇELA~—ZM* The number /,,(*f*, ^4)=inf M(A) where the infimum is taken over all majorants of *f* on A is called the upper integral of *f* over. 4. Proposition 1. Let A £0* and let f be a function on A". Iff^O then Iu(f,A)^0. Proposition 2. Let A Go- and let f be a junction on A~. Then the function!,, (*f*, •) is additive on a A* Proposition 3. Let A^a and let {*f*,,} be an increasing sequence of functions defined on A". If Iu(Ju A) > — 00 then lim Iu(fn, A) = 7tt(lim *f*,,, A). Definition. Let A £0^ and let *f* be a function on A". Choose dis-joint sets Au • • •, An from a such that U?ml Ai —A and set Iu(f,A)= itluiffAi). When /,, (*f*, i4) = —/,, (—*f*, i4) 5^ ± 00 this common value is called the integral of *f* over -4 and it is denoted by / (*f*, A). By [12, (1.1)] the sets Au • • •, ^4n always exist and by Proposi-tion 2 the value of *f*« (*f*, i4) is independent of their choice. Thus the previous definition extends the original meaning of the upper integral. If AÇz<r*, ty (A) denotes the family of all functions *f* defined on A" for which the integral I (f, A) exists and ^o (A) = {fe$(A):\f\ey (A)}. Theorem 1. If A G*~ and/G^ (-4), then fG^ (B) for every J5G^. Theorem 2. Let AÇz<r*, *f*, gGty (A) and let a, b be real numbers. If h(x) = af (x) + bg (x) for all xÇzA~ for which af(x) + bg (x) has meaning, then AG^J (-4) and I (h, A) = aI (f, A) + bI (g, A). Moreover, iff, gG^o (A) then also hÇzS$0(A). Corollary. LetAÇiaandf, gGtyo (A). Then max (/, g) G^o (A) and min (f, g) G$o (A). Moreover, if hE$ (A) andf^h^g, then hE$0 (A). THEOREM 3. Let A (E<r, *f*, g> hnG^ (A), and let f^hn^g for n = 1, 2, • • •. If lim hn = h then hEW (A) and I (h, A) = lim I (hn, A). EXAMPLE 1. Let P be r-dimensional Euclidean space and let a be the system of all half-open intervals (bounded, unbounded, or de-generate). For xÇzP~ let KX consist of all sequences {Kn} n"=1Co" which satisfy condition 5C2 and such that either #Efl*-i Kn °r Kn = 0 for all sufficiently large n. Then the convergence K satisfies conditions 3Ci-3C5 (see [l2, 3.2]) and the integral I coincides with the integral defined by Maffk in [7]. In particular, if r = 1 and if G is the restric-tion of the Lebesgue measure, then the integral I coincides with the classical Perron integral (see [8] or [18]). EXAMPLE 2. Let P be the discrete space of positive integers, let <r consist of the empty set, singletons (n), and intervals [n, +<*>), n = 1, 2, • • •, and let K = K° be the natural convergence. If G is the counting measure on <r, then *f* £ *$(P) if and only if the series] C"=i f (n) is conditionally convergent; I (J, P) = X)*-i f (n) when either side has meaning. Given another convergence K' = {KI: #£P~} which satisfies con-ditions JC1-JC5, we can introduce the symbols V and $' the meaning of which is obvious. The connection between I, ^3 and I', ty' is given by the following proposition. Proposition 4. Let KX' CKX for all x (EP~. Then for every A £0-, ty (A) CW (A) and I (f, A) =I' (f, A) for allf^^A). A point xÇzP~ is said to be simple if and only if Tx= {Un) *-i, o"ax = <ruv ancl f°r every {Bn} and {Cn} from KX (<JU^ also {Bn\JCn} and {Bn — Cn} belong to Kx (aux). Notice that if <T = a* and K = K° is the natural convergence, then #£P" is simple whenever r* = {Z7n} *ml. Theorem 4. Let A £<r* and let f be a function on A". Let x£P~ be a simple point at which P' is Hausdorff5 and let either x^A" or # (-<?) (*, A)^0. Further suppose that f'£$ (£) for every BE<r2 for which x (£B~ and that UmI (f, A - Bn) = c j* ± 00 for every {#«} £** for which x$: (A—Bn) ~', n = l, 2, • • •. 77& ew For the remainder of this paper we shall assume that P is a locally compact Hausdorff space. Unless otherwise specified our terminology concerning measures is that of [3]. If A QP, XA denotes the character-istic function of A. Let SE be the family of all sets AQP for which XAdc E$o (P) for every compact set CQP. For 4G2 we let I (A) -/.eu, p). Theorem 5. The triple (P, £, t) is a complete measure space and % contains all open subsets of P. The measure t is inner regular on open sets and outer regular and finite on compact sets. Moreover, if CC.P is compact, then L(C) = Inf G (A) where the infimum is taken over all sets AÇ^a* whose interior contains C.6 THEOREM 6. A f unction f belongs to $o (P) if and only if the Lebesgue integral fpfdi exists; I (f, P) —fpfdt, when either side has meaning. More-over, if f or every A Çz<r and for every f unction f on A" the upper integral *f*,, (*f*, A) is independent of values which f takes on A"—A, then <rQ% and a function f belongs to tyo (A), A £<r, if and only if the Lebesgue integral JAfdi exists; I (J, A) —JAfdi when either side has meaning. For AQP let io (A) =inf t (£7), where the infimum is taken over all open sets UQP for which AQU. It is easy to see that t0 is an outer measure in P and we shall denote by Xo the family of all to-measurable subsets of P. Theorem 7. The triple (P, %0, t0) is a complete measure space and the measure t0 is regular.7 Moreover, if (P, %, jx) is a measure space with a regular measure ix, a (Z%, and G (A) —n (A) for every A Ç. <rfor which A" is compact, then %Q%Q and n (A) =toC4) for every A E3L Theorem 8. Always XoQX and t0 (^4) =t (^4) for every set -4££0 which is Lo-a-finite. If P is paracompact then LO (A) =^ (, (A) for every AGXo. Theorem 9. Suppose

that aCXo and that G (A) –iQ (A) for every A £cr for which A" is compact. Let A £<r and let f be a function on A" for which the Lebesgue integral fAfdto exists. If A"-A is i^cr-finite then fe$o (A) and I (f, A) =JAfdu>. Corollary. Let <rC£o and let G (A) =i0 (^4) for every A Çiafor which A~ is compact. If i0 is a-finite then for every A £(r and for every f unction f on A~~ the upper integral Iu (f, A) is independent of values which f takes on A~-A. Let K = K° be the natural convergence. It was proved in [17] that in this case (P, %, i) = (P, Xo, to) and every function *f* £ ty (A) is t-measurable. Moreover, if <r = <r* then ty (A) =$oC4) for every A GO". Example 3. Let P = (-00, +00) and let a consist of all intervals which either do not contain 0 in their closures or are of the form (-€, e) where €>0. Let KX = K% for x9*0 and let K0 consist of the net {(-€, €)} €>o and the trivial net {0}. Finally, let G be the restriction of the Lebesgue measure. Then K satisfies conditions 3Ci-3C6 and *f* belongs to $o (P) if and only if the Lebesgue integral JtZf exists; *f* (*f*, P) =ftZf when either side has meaning. On the other hand, *f* belongs to $(P) if and only if the Cauchy principle value *f*+00 / /» —€ /» +00 \ / = lim (*f*+ *f*) exists; / (*f*, P) = (pv) ftZf when either side has meaning. Notice that 0 is not a simple point of P. Complete proofs of the results stated here will be given in [13], [14] and [16].

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