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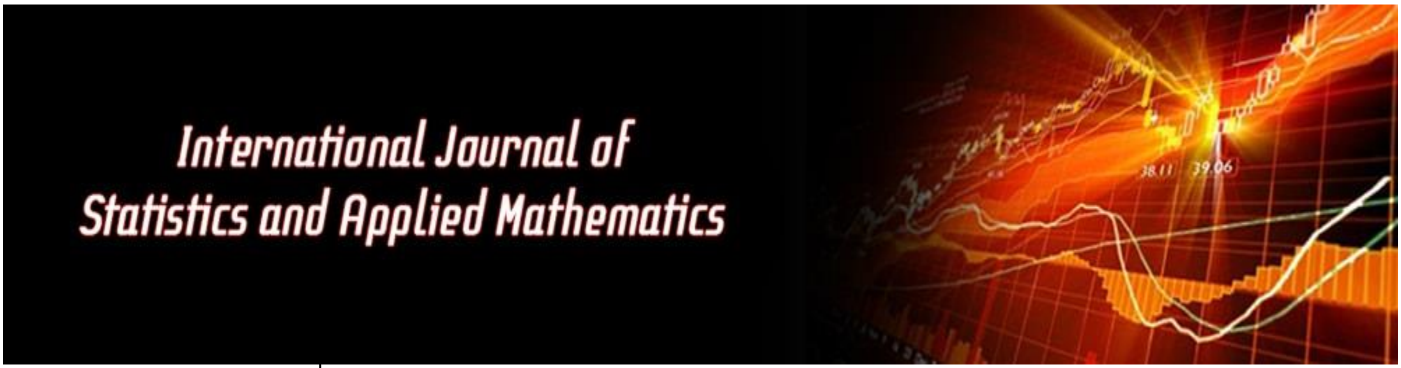

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K-l analytic functions and their related theorems

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Abstract
In this paper, we present the concept of K-l analytic functions on the complex plane and discuss the properties of the K-l analytic functions. Then we give the expression of K-l analytic functions and proved the Cauchy integral formula of K-l analytic functions. Also, the power series expansion theorem and the Fourier series of K-l analytic function were proved in this study.

Keywords: K-l analytic functions, cauchy integral formula, power series, fourier series

1. Introduction

Analytic functions have been applied in many fields as a powerful tool, e.g., theoretical physics, astromechanics, fluid mechanics and elastic mechanics. Over the past two hundred years, systematic theories in analytic functions have been established through the efforts of many scholars [1-5]. Recently, the concept of K-analytic function have been proposed and its analytical properties have been studied by some researchers (see [6-9]). In [6-7], the authors concluded a necessary and sufficient conditions of K-analytic function, and deducts the relation between K-analytic function and K-integral, the relation between K-analytic function and K-harmonic function. Zhang jianyuan and his collaborators [8] used the series theory to propose the power series expansion of the K-analytic function and its zero isolation and uniqueness. Then, in [9], the Fourier series of K-analytic function is given and proved. In [10], the authors presented the concept of K- bianalytic function. The function $f(z)$ that satisfies the condition $\frac{\partial^2 f}{\partial \bar{z}^2(k)} = 0$ is called the K- bianalytic function in the complex plane. The K-bianalytic function expression and analysis properties are given, and its series forms are proved. Then the authors [11] explained the concept of K- trianalytic functions, and discussed the properties and expression and the power series expansion of the K- trianalytic functions.

In this research, we consider the more general case of homogeneous higher-order complex equations, namely K-l analytic functions, which contain K-analytic function classes and K-bianalytic analytic function classes. We gave the K-l analytic functions expression. And we proved Cauchy integral formula of K-l analytic function. In the last part, we discussed the power series expansion and the Fourier series expansion of K-l analytic functions.

2. Preparatory knowledge

Definition 1 [6] (definition of K- Complex number) Let function $f(z)$ be defined in the region D, we say that the fo rm $z(k) = x + iky(k \in R, k \neq 0)$ is k- Complex number of $z = x + iy$.

Definition 2 [6] (definition of K-derivative) Let function $f(z)$ be defined in a neighborhood of z_0, $z(k) = x + iky(k \in R, k \neq 0)$, if the limit $\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z(k)} = \lim_{\Delta z \to 0} \frac{f(z) - f(z_0)}{z(k) - z_0(k)}$ is exist, then we call function $f(z)$ is k-differentiable at z_0, and this limit is defined as the k-derivative of function $f(z)$ at z_0, denoted as $f'_{(k)}(z_0)$ or $\left.\frac{df(z)}{dz(k)}\right|_{z=z_0}$.

Lemma 1 [6] (Necessary and sufficient conditions of K-analytic) the function $f(z) = u(x, y) + iv(x, y)$ is K-analytic in the region D, $\Leftrightarrow u, v$ is differentiable in D, and the condition of C-R-K holds, where condition C-R-K is : $\begin{cases} u_x = v_y/k \\ v_x = -u_y/k \end{cases}$, $(u(x,y), v(x,y)$ are binary real functions).

Let $\frac{\partial}{\partial \bar{z}(k)} = \frac{\partial}{\partial x} + \frac{i}{k}\frac{\partial}{\partial y}$, where $\bar{z}(k)$ is the complex conjugate of $z(k)$. Then the complex form of function C-R-K is $\frac{\partial f}{\partial \bar{z}(k)} = 0$ [10].

Lemma 2 [8] (power series expansion of K-analytic) Suppose function $f(z)$ is K-analytic in the region D, and $B(k): |(z-a)(k)| < R \subset D$, we have

$$f(z) = \sum c_n (z-a)^n(k), z \in B(k),$$

Where

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}(k)} d\zeta(k) = \frac{f^n(a)}{n!},$$

$\left(\Gamma_\rho: |(z-a)(k)| = \rho, 0 < \rho < R, n = 0,1,2,\cdots\right)$, and the expansion is unique.

3. K-l analytic functions and their related theorems

Definition 3 (definition of K-l analytic function) Let D is a region on the complex plane, the function $f(z)$ is a complex function defined in the region D and has a l-th derivative $\frac{\partial^l f}{\partial \bar{z}^l(k)}, (l = 1,2,3\cdots, \bar{z}(k) \in D)$. If a given function $f(z)$ satisfies

$$\frac{\partial^l f}{\partial \bar{z}^l(k)} = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial^{(l-1)} f}{\partial \bar{z}^{(l-1)}(k)}\right) = 0,$$

then $f(z)$ is called K-l analytic function in D. We use $F(D_l(k))$ to represent the set of all K-l analytic functions in D. In particular, when $l = 1$, $f(z)$ is the K-analytic function defined in text [6]. When $l = 2$, $f(z)$ is the K- bianalytic function defined in [10].
For the above definition we have the following four propositions.

Proposition 1 If $f(z) \in F(D_l(k))$, then $\frac{\partial f}{\partial \bar{z}(k)} \in F(D_{l-1}(k))$, and vice versa.

Proposition 2 If $f(z) \in F(D_l(k))$, then $\frac{\partial^{(l-1)} f}{\partial \bar{z}^{2(l-1)}(k)} \in F(D_1(k))$, and vice versa.

Proposition 3 Both set of the K-analytic functions and set of the K- bianalytical functions are a subset of the K-l analytic functions, that is$F(D_1(k)) \subset F(D_2(k)) \subset \cdots \subset F(D_l(k)) \subset \cdots$.

Proposition 4 If function $f(z) \in F(D_l(k))$, and $\varphi(z)$ is an arbitrary K- analytic function, then $f(z)\varphi(z)$ is K-l analytic in D.

Proof : Under assumptions we can know $\frac{\partial^l f}{\partial \bar{z}^l(k)} = 0$, and $\frac{\partial \varphi}{\partial \bar{z}(k)} = 0$， we conclude that

$$\frac{\partial}{\partial \bar{z}(k)}\big(f(z)\varphi(z)\big) = \frac{\partial f}{\partial \bar{z}(k)}\varphi(z) + f(z)\frac{\partial \varphi}{\partial \bar{z}(k)} = \frac{\partial f}{\partial \bar{z}(k)}\varphi(z).$$

Thus there is

$$\frac{\partial^2}{\partial \bar{z}^2(k)}\big(f(z)\varphi(z)\big) = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial f}{\partial \bar{z}(k)}\varphi(z)\right) = \frac{\partial^2 f}{\partial \bar{z}^2(k)}\varphi(z),$$

And

$$\frac{\partial^3}{\partial \bar{z}^3(k)}\big(f(z)\varphi(z)\big) = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial^2 f}{\partial \bar{z}^2(k)}\varphi(z)\right) = \frac{\partial^3 f}{\partial \bar{z}^3(k)}\varphi(z) + \frac{\partial^2 f}{\partial \bar{z}^2(k)}\frac{\partial \varphi}{\partial \bar{z}(k)} = \frac{\partial^3 f}{\partial \bar{z}^3(k)}\varphi(z)$$

$$\frac{\partial^l}{\partial \bar{z}^l(k)}\big(f(z)\varphi(z)\big) = \frac{\partial^l f}{\partial \bar{z}^l(k)}\varphi(z) = 0.$$

Then we can know that $f(z)\varphi(z)$ is K-l analytic in D.

Theorem 1 (expansion of K-l analytic functions) If function $f(z) \in F(D_l(k))$, then the following is established

$$f(z) = \sum_{m=0}^{l-1} \bar{z}^m(k) \cdot \varphi_m(z), \tag{1}$$

where $\varphi_m(z)(m = 0,1,2,\cdots,l-1)$ is arbitrary K-analytic function in the region D.

Proof: (*Mathematical induction*)
Base case: When $l = 1, \frac{\partial f(z)}{\partial \bar{z}(k)} = 0$. Show that $f(z)$ is arbitrary k - analytic function. Formula (1) is easily seen to be true.
When $l = 2$, $f(z)$ is the k- bianalytic analytic function in reference [10], whose expression has been proved to be

$$f(z) = \bar{z}(k)\varphi_1(z) + \varphi_0(z),$$

$\varphi_0(z), \varphi_1(z)$ are K- analytic functions in D

Inductive step: Assume formula (1) holds, for $l = n, \frac{\partial^n f(z)}{\partial \bar{z}^n(k)} = 0$, i.e.,

$$f(z) = \bar{z}^{n-1}(k)\varphi_{n-1}(z) + \bar{z}^{n-2}(k)\varphi_{n-2}(z) + \cdots + \varphi_0(z)$$

Then if $l = n + 1$, then $\frac{\partial^{n+1} f(z)}{\partial \bar{z}^{n+1}(k)} = \frac{\partial}{\partial \bar{z}(k)}\left(\frac{\partial^n f(z)}{\partial \bar{z}^n(k)}\right) = 0$. Indicate that $\frac{\partial^n f(z)}{\partial \bar{z}^n(k)}$ is a K - analytic function.

Let

$$\varphi_n^*(z) = \frac{\partial^n f(z)}{\partial \bar{z}^n(k)}, \tag{2}$$

we write $f_1(z) = \frac{1}{n!}\bar{z}^n(k) \cdot \varphi_n^*(z)$. Then $\frac{\partial^{n+1} f_1(z)}{\partial \bar{z}^{n+1}(k)} = 0$, i.e., $(f_1(z))$ is the K-(n+1) analytic function, and there are

$$\frac{\partial^n f_1(z)}{\partial \bar{z}^n(k)} = \varphi_n^*(z) \tag{3}$$

Subtracting (3) from (2) is obtained

$$\frac{\partial^n\big(f(z) - f_1(z)\big)}{\partial \bar{z}^n(k)} = 0,$$

So $\big(f(z) - f_1(z)\big)$ is the K-n analytic function. Using the induction hypothesis, we can get:

$$f(z) = \frac{1}{n!}\bar{z}^n(k)\varphi_n^*(z) + \bar{z}^{n-1}(k)\varphi_{n-1}(z) + \bar{z}^{n-2}(k)\varphi_{n-2}(z) + \cdots + \varphi_0(z)$$

$$= \bar{z}^n(k)\varphi_n(z) + \bar{z}^{n-1}(k)\varphi_{n-1}(z) + \bar{z}^{n-2}\bar{z}(k)\varphi_{n-2}(z) + \cdots + \varphi_0(z),$$

where $\varphi_n(z) = \frac{1}{n!}\varphi_n^*(z)$. Thereby showing that indeed formula (1) holds.

Theorem 2 (integral formula of K-l analytic functions) Let D be the region bounded by a closed curve C, if function $f(z)$ is K-l analytic in D, and $f(z), \frac{\partial f}{\partial \bar{z}(k)}$ are continuous on $\bar{D} = D + C$, then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)(k)} d\zeta(k) - \frac{1}{2\pi i}\sum_{m=1}^{l-1}\sum_{j=0}^{l-m-1}\frac{(-1)^j}{m!\,j!}\int_C \frac{\bar{z}^j(k)\cdot\left(\overline{\zeta(k)}^m - \overline{z(k)}^m\right)}{(\zeta - z)(k)}\frac{\partial^{m+j} f}{\partial \bar{z}^{m+j}(k)} d\zeta(k), \tag{4}$$

where $z \in D$.

Proof: According to formula (1) and Cauchy integral formula of K- analytic function [7], we have

$$f(z) = \sum_{m=0}^{l-1} \bar{z}^m(k) \cdot \varphi_m(z) = \sum_{m=0}^{l-1} \bar{z}^m(k) \cdot \frac{1}{2\pi i}\int_c \frac{\varphi_m(\zeta)}{(\zeta - z)(k)} d\zeta(k)$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{l-1}\int_c \frac{\bar{z}^m(k)\varphi_m(\zeta)}{(\zeta - z)(k)} d\zeta(k)$$

$$=\frac{1}{2\pi i}\int_{c}\frac{\varphi_0(\zeta)}{(\zeta-z)(k)}d\zeta(k)+\frac{1}{2\pi i}\sum_{m=1}^{l-1}\int_{c}\frac{\bar{z}^m(k)\varphi_m(\zeta)}{(\zeta-z)(k)}d\zeta(k)$$

$$=\frac{1}{2\pi i}\int_{c}\frac{\varphi_0(\zeta)}{(\zeta-z)(k)}d\zeta(k)+\frac{1}{2\pi i}\sum_{m=1}^{l-1}\int_{c}\frac{\bar{\zeta}^m(k)\varphi_m(\zeta)}{(\zeta-z)(k)}d\zeta(k)-\frac{1}{2\pi i}\sum_{m=1}^{l-1}\int_{c}\frac{\bar{\zeta}^m(k)-\bar{z}^m(k)}{(\zeta-z)(k)}\varphi_m(\zeta)d\zeta(k)$$

$$=\frac{1}{2\pi i}\int_{c}\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)-\frac{1}{2\pi i}\sum_{m=1}^{l-1}\int_{c}\frac{\bar{\zeta}^m(k)-\bar{z}^m(k)}{(\zeta-z)(k)}\varphi_m(\zeta)d\zeta(k).$$

On the one hand, we take the mth, $(m+1)$th, $\cdots l$th derivative of both sides of equation (1) with respect to $\bar{z}$, we have

$$\varphi_m(\zeta)=\frac{1}{m!}\sum_{j=0}^{l-m-1}\frac{(-1)^j\bar{z}^j(k)}{j!}\cdot\frac{\partial^{m+j}f}{\partial\bar{z}^{m+j}(k)},m=1,2,\cdots l-1.$$

Then the formula (4) is established.
By Theorem 1, we can get the power series expansion theorem of K-l analytic functions.

Theorem 3 (Taylor series of K-l analytic functions) If $f(z)$ is a K-l analytic function in the region D, and $\mathrm{B}(k)\colon|(z-a)(k)|<R\subset D$, for $\forall a\in D$, then $f(z)$ can be expanded into power series in D

$$f(z)=\sum_{n=0}^{+\infty}\sum_{m=0}^{l-1}c_{nm}\bar{z}^m(k)\cdot(z-a)^n(k),z\in B(k), \tag{5}$$

where

$$c_{nm}=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi_m}{(\zeta-a)^{n+1}(k)}d\zeta(k),\ \ (n=0,1,2\cdots), \tag{6}$$

$(\Gamma_\rho\colon|(z-a)(k)|=\rho,0<\rho<R,n=0,1,2,\cdots)$, and the expansion is unique.

Proof: According to theorem 1, we have (1), where $\varphi_m(z),m=0,1,2,\cdots,l-1$, is a K-analytic function in D. The following formula obtained by the power series expansion of the K-analytic function (Lemma 2)

$$\varphi_m(z)=\sum_{n=0}^{+\infty}c_{nm}(z-a)^n(k),z\in B(k),m=0,1,2,\cdots,l-1,$$

where $c_{nm}=\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi_m}{(\zeta-a)^{n+1}(k)}d\zeta(k)$, $(n=0,1,2\cdots)$.

So, we have

$$f(z)=\sum_{n=0}^{+\infty}\sum_{m=0}^{l-1}c_{nm}\bar{z}^m(k)\cdot(z-a)^n(k),z\in B(k).$$

It is obvious that the expansion (5) is unique from the uniqueness of the power series expansion of the K-analytic function. Theorem is proved.

Theorem 4 (Fourier series of K-l analytic functions) Suppose function $f(z)$ is K-l analytic in the region D, and $\mathrm{B}(k)\colon|(z-a)(k)|<R\subset D,(\forall a\in D)$, then we have

$$f(z)=\sum_{m=0}^{l-1}\frac{\rho^m}{2}(\alpha_{0m}+i\beta_{0m})(\cos m\theta-i\sin m\theta)+\sum_{n=1}^{+\infty}\sum_{m=0}^{l-1}\rho^m(\alpha_{nm}+i\beta_{nm})(\cos(n-m)\theta+i\sin(n-m)\theta).$$

Where

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\rho,\theta)d\theta$$

$$\beta_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,\theta)d\theta$$

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\sin n\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\cos n\theta\, d\theta\,, (n = 1,2,\cdots)$$

$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\sin n\theta\, d\theta\,, (n = 1,2,\cdots)$$

and the expansion is unique.

Proof: Because function $f(z)$ is K-l analytic in D, form (5) and (6) are established. In $\Gamma_\rho\colon |(z-a)(k)| = \rho$, suppose $(z-a)(k) = \rho e^{i\theta}(0<\rho<R)$, then $(z-a)^n(k) = \rho^n e^{in\theta}, dz(k) = i\rho e^{i\theta}d\theta$, thus（6）is

$$c_{nm} = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\varphi_m}{(z-a)^{n+1}(k)}dz(k) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\varphi_m}{\rho^{n+1}e^{i(n+1)\theta}}i\rho e^{i\theta}d\theta$$

$$= \frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}\varphi_m e^{-in\theta}d\theta \tag{7}$$

When $n \geq 1$, we can see that $\varphi_m \cdot z^{n-1}$ is K- analytic in D, we have $\int_{\Gamma_\rho}\varphi_m z^{n-1}dz(k) = 0$, i.e., $\int_{-\pi}^{\pi}\varphi_m i\rho^n e^{in\theta}d\theta = 0$, and so we obtain

$$0 = \frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}\varphi_m i e^{in\theta}d\theta. \tag{8}$$

Add form (7) to form (8), we have

$$c_{nm} = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi}\varphi_m \cos n\theta\, d\theta. \tag{9}$$

Subtracting (7) from (8) is obtained.

$$c_{nm} = \frac{-i}{\pi\rho^n}\int_{-\pi}^{\pi}\varphi_m \sin n\theta\, d\theta. \tag{10}$$

Let $\varphi_m = u_m(\rho,\theta) + iv_m(\rho,\theta)$, by (7'), (8') we have

$$c_{nm}\rho^n = \frac{1}{\pi}\int_{-\pi}^{\pi}u_m(\rho,\theta)\cos n\theta\, d\theta + \frac{i}{\pi}\int_{-\pi}^{\pi}v_m(\rho,\theta)\cos n\theta\, d\theta,$$

$$c_{nm}\rho^n = \frac{1}{\pi}\int_{-\pi}^{\pi}v_m(\rho,\theta)\sin n\theta\, d\theta + \frac{-i}{\pi}\int_{-\pi}^{\pi}u_m(\rho,\theta)\sin n\theta\, d\theta.$$

Let $c_{nm}\rho^n = \alpha_{nm} + i\beta_{nm}$, then

$$\alpha_{nm} = \frac{1}{\pi}\int_{-\pi}^{\pi}u_m(\rho,\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}v_m(\rho,\theta)\sin n\theta\, d\theta,$$

$$\beta_{nm} = \frac{1}{\pi}\int_{-\pi}^{\pi}v_m(\rho,\theta)\cos n\theta\, d\theta = -\frac{1}{\pi}\int_{-\pi}^{\pi}u_m(\rho,\theta)\sin n\theta\, d\theta.$$

When $n = 0$, directly derived from (7) formula

$$\alpha_{0m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_m(\rho,\theta)d\theta,$$

$$\beta_{0m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_m(\rho,\theta)d\theta.$$

Obviously α_{nm}, β_{nm} are Fourier coefficient of $u_m(\rho,\theta), v_m(\rho,\theta)$. Substituting α_{nm}, β_{nm} in (5) formula,

$$f(z) = \sum_{n=0}^{+\infty}\sum_{m=0}^{l-1} c_{nm}\bar{z}^m(k)\cdot(z-a)^n(k)$$

$$= \sum_{n=0}^{\infty}\left(c_{n0} + c_{n1}\bar{z}(k) + \cdots + c_{n(l-1)}\bar{z}^{l-1}(k)\right)(z-a)^n(k)$$

$$= \sum_{n=0}^{\infty}\left(c_{n0} + c_{n1}\rho e^{-i\theta}\bar{z}(k) + \cdots + c_{n(l-1)}\rho^{l-1}e^{-i(l-1)\theta}\right)\rho^n e^{in\theta}$$

$$= \sum_{n=0}^{\infty}\left(c_{n0}\rho^n e^{in\theta} + c_{n1}\rho^{n+1}e^{i(n-1)\theta}\bar{z}(k) + \cdots + c_{n(l-1)}\rho^{n+l-1}e^{i(n-l+1)\theta}\right)$$

$$= \sum_{n=0}^{\infty}(\alpha_{n0} + i\beta_{n0})e^{in\theta} + \sum_{n=0}^{\infty}\rho(\alpha_{n1} + i\beta_{n1})e^{i(n-1)\theta} + \cdots + \sum_{n=0}^{\infty}\rho^{l-1}\left(\alpha_{n(l-1)} + i\beta_{n(l-1)}\right)e^{i(n-l+1)\theta}$$

$$= \sum_{m=0}^{l-1}\frac{\rho^m}{2}(\alpha_{0m} + i\beta_{0m})(\cos m\theta - i\sin m\theta) + \sum_{n=1}^{+\infty}\sum_{m=0}^{l-1}\rho^m(\alpha_{nm} + i\beta_{nm})(\cos(n-m)\theta + i\sin(n-m)\theta).$$

The uniqueness is obvious.

4. References

1. Jianding Wang. Semi-analytic function conjugate analytic function, M. Beijing University of Technology Press, 1988.
2. Zhao Zhen. Tri-analytic functions, complex harmonic functions and their basic boundary value problem. J Journal of Beijing Normal University, Natural Sci. 1995; 31(2):175-179.
3. Wang Minghua. Some properties of tri-analytic functions and its Hilbert boundary value problem. J Journal of Beijing Normal University, Natural Sci. 1998; 34(1):13-20.
4. Yang Pi-wen. The k-regular Function and Some Boundary Value Problems. J Journal of Sichuan Normal University, Natural Sci. 2001; 24(1):5-8.
5. Rckya HH. Generalized analytic function, M, Beijing: people's education Press, 1960.
6. Zhang Jianyuan. K-analytic function and its existence condition. J Journal of Yunnan University of Nationalities: Natural Science Edition. 2007; 16(4):298-302.
7. Zhang Jianyuan, Zhang Yimin, Jiang Ruiwu, Liu Chengping. K-integral of complex functions. J Journal of Yunnan Normal University: Natural Science Edition. 2009; 29(1):24-28.
8. Zhang Jianyuan, Zhang Yimin, Jiang Ruiwu, Liu Chengping. Expansion of K-analytic function in Power series. J Journal of Dali University. 2009; 8(4):14-18.
9. Hong Li, Hongkui Li. Fourier series of k-analytic functions. J International Journal of Trend in Research and Development. 2018; 5(5):318-319.
10. Hong Li, Tongbo Liu. K-bianalytic functions and their related theorems. J International Journal of Statistics and Applied Mathematics. 2018; 3(6):36-40.
11. Hong Li, Tongbo Liu. K-trianalytic functions. J International Journal of Statistics and Applied Mathematics. 2019; 4(1):01-04.