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# η- Ricci solitons on lorentzian para-sasakian manifolds defined with $W_2$−curvaturetensor

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**Abstract**
In the present paper $\eta$- Ricci solitons on Lorentzian Para- Sasakian manifolds satisfying $(\xi,.)s.W_2 = 0$ and $(\xi,.)\ W_2.S = 0$ are treated. The results obtained on $\eta$- Ricci solitons on para- Kenmotsu manifolds have motivated us to investigate $\eta$- Ricci solitons on Lorentzian Para-Sasakian Manifolds satisfying the same conditions and quasi-similar results have been obtained. In fact, we have proved that Lorentzian Para-Sasakian manifolds satisfying $(\xi,.)\ s.W_2 = 0$ and having $\eta-$ Ricci soliton structure are Einstein or quasi-Einstein manifolds according to the value $\mu$ and $\lambda$. The same results have been proved on Manifolds satisfying $(\xi,.)\ W_2.S = 0$.

**Keywords:** Ricci solitons, $\eta-$ricci solitons, lorentzian para-sasakian manifolds, $W_2$ curvature tensor

## 1. Introduction

### 1.1 Background

In last years the Ricci solitons have interested most of geometers as topic of study on different manifolds. Contact and para-Contact have been among the most considered in the study of those solutions of Ricci flows. The interest in study has considerably motivated by the recent Perelman proof of Poincar´e Conjecture using Ricci flows. The Ricci Flows have been introduced by R.S as generalization of Einstein metrics in 1982 [1]. Ricci flow is an evolution equation of heat equation type for the metric on Reimannian manifold and is
defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2Sij \qquad (1.1)$$

### 1.2 Ricci soliton

A Ricci soliton $(g, v, \lambda)$ on Reimannian manifold $(M, g)$ is given by
$L_V g + 2S + 2\lambda g = 0$ (1.2)
Where S is the Ricci tensor, $L_V$ is the Lie derivative operator on M in direction V and $\lambda$ is a scalar. Ricci solitons have been discussed on different manifolds such us: K¨ahler manifolds [2], On Para - Sasakian manifolds [3], on Kenmotsu [4] and f -Kenmotsu Manifolds [5] and more manifolds have been discussed by various authors. A general notion of $\eta-$ Ricci soliton has been introduced by J.T. Chao and M. Kimura in [6] and has been studied on Hopf hypersurfaces in complex spaces by C. C˘alin in [7]. In this paper we consider $\eta-$ Ricci solitons on Lorentzian para- Saasakian manifolds satisfying $(\xi,.)$ s.W8 = 0 and $(\xi,.)$W8.S = 0. Note that Blaga [8] has obtained some results on Ricci solitons satisfying $(\xi,.)$ S.R = 0 and $(\xi,.)$ R.S = 0 on the same manifolds.

## 2. Lorentzian para-kenmotsu manifolds

If M is m- dimensional differentiable manifold, $\phi$ a $(1, 1)-$type tensor field, $\eta$ $1-$form and g a Lorentzian metric on M , then
Deftnition 2.1. [8] $(\phi, \xi, \eta, g)$ is said to be Lorentzian structure on M if:

$$\phi\xi = 0, \quad \eta(\phi) = 0, \qquad (2.1)$$

$\eta(\xi) = -1, \quad \phi 2 = I + \eta \otimes \xi,$ (2.2)

$g(\phi., \phi.) = g + \eta \otimes \eta,$ (2.3)

$(\nabla_X \phi)Y = g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X \quad \forall X, Y \in X(M)$ (2.4)
where X(M) is algebra of vector fields on M and ∇ is the Levi- Civita connection associated to g. and
$\eta \otimes \eta(X, Y)$ stands for $\eta(X)\eta(Y)$

Using (2.3) and (2.1), we get

$g(\xi, \xi) = -1,$ (2.5)

$\eta(X) = g(X, \xi)$ (2.6)

and

$g(\phi X, Y) = g(X, {}_{\phi}Y)$ X, Y ∈ X(M) (2.7)

The immediate properties of Lorentzian structure are given in the following proposition proved in [8]
Proposition 2.2. On Lorentzian Para- Sasakian manifolds, the followings hold:

$\nabla_X \xi = \phi X,$ (2.8)

$\eta(\nabla_X \xi) = 0, \ \nabla \xi \xi = 0,$ (2.9)

$R(X, Y)\xi = -\eta(X)Y + \eta(Y)X$ (2.10)

$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(y)g(X, Z), \quad \eta(R(X, Y)\xi) = 0$ (2.11)

$(\nabla_X \eta)Y = (\nabla_Y \eta)X = g(\phi_X, Y), \quad \nabla\xi\eta = 0,$ (2.12)

$L\xi\phi = 0, L\xi\eta = 0, L\xi g = 2g(\phi.,.)$ (2.13)

Where R is Reimannian curvature tensor field and ∇ is Levi- Civita connection associated to g.
From this proposition we get that $g(X, {}_{\phi}Y)$ is symmetric and

$(\nabla g(X, \phi)) (Y, Z) = \eta(Y)(X, Z) + \eta(Z)g(X, Y) + 2\eta(X)\eta(Y)\eta(Z)$ (2.14)

and

$g(\phi X, \phi^2 Y) = g(X, {}_{\phi}Y)$ (2.15)

Let (M, ϕ, ξ, η, g) be a Lorentzian manifold. The data satisfying the equation

$L\xi g + 2s + 2\lambda g + 2\mu\eta \otimes \eta$ (2.16)

Where Lξ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g, and λ and μ are scalars is said to be η−Ricci soliton on M. Writing Lξ in terms of the Levi- Civita connection (2.16) becomes:

$2S (X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - \mu\eta(X)\eta(Y) \ \forall X, Y \in X(M).$ (2.17)

Using (2.8) in (2.16), we get

$S(X, Y) = -g({}_{\phi}X, Y) - \lambda g(X, Y) - \mu\eta(X)\eta(Y).$ (2.18)

In [9] Matsumoto, Koji and Mihai proved that on Lorentzian para- Sasakian manifold (M, ϕ, ξ, η, g) the Ricci tensor satisfies

$S(X, \xi) = (\dim(M) - 1))\eta(X)$ (2.19)

and

$S(\phi X, {}_{\phi}Y) = S(X, Y) + (\dim(M) - 1)\eta(X)\eta(Y)$ (2.20)

Putting Y = ξ in (2.18), we get

$$\mu - \lambda = 2n \quad (2.21)$$

for a $(2n + 1)-$dimensional Lorentzian Para- Sasakian manifold M. In [8], the author studied the Lorentzian Para - Sasakian Manifolds having cyclic Ricci tensor and those with cyclic $\eta-$ recurrent Ricci tensor and proved that there is no Ricci soliton with potential vector field $\xi$. In the same paper the author discussed the Lorentzian Para- Sasakian manifolds satisfying $(\xi,.)$ R.S = 0, and $(\xi,.)$ S.R = 0 and proved that in such cases M is Einstein. In the present paper Lorentzian Para- Sasakian satisfying $(\xi,.)$ $W_2$.S and those satisfying $(\xi)$ S.$W_2$ are discussed.
Note that $W_2$ curvature tensor has been introduced by by Pokhariyal and Mishra [10] and it is defined as

$$W_2(X, Y) Z = R (X, Y) Z + \frac{1}{2n} (g(X, Z)QY - S(Y, Z)QX) \quad (2.22)$$

for a $(2n + 1)-$ Lorentzian Para- Sasakian Manifolds, where QX is Ricci tensor operator defined as

$$g(QX, Y) = S(X, Y). \quad (2.23)$$

Using 2.18, we get

$$QX = -\phi X - \lambda X - \mu\eta(X)\xi. \quad (2.24)$$

The following definition will be used later in derivation of our results.
Deftnition 2.3. A Ricci soliton is said to be quasi- Einstein if its Ricci curvature tensor field S is linear combination (with scalars $\lambda$ and $\mu$ respectively, with $\mu$ different of zero) of g and the tensor product of a non- zero 1- form $\eta$ satisfying $\eta(x) = g(X, \xi)$, for $\xi$ a unit vector field and respectively, Einstein if S is collinear with g.

**3. $\eta-$Ricci Solitons on Lorentzian Para- Sasakian Manifolds satisfying $(\xi,.)$ $W_2$.S = 0.**
The condition to be satisfied by S is

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0 \quad (3.1)$$

$$S(W_2(\xi, X)Y, Z) = \eta(Y)\{g(\phi X, Z) + \lambda g(X, Z) - \mu\eta(X)\eta(Z)$$

$$+ \frac{1}{2n}[g(X, Z) + \eta(X)\eta(Z) + 2\lambda g(\phi X, Z) + \lambda 2g(X, Z) + 2\lambda\mu\eta(X)\eta(Z) \quad \mu 2\eta(X)\eta(Z)]. \quad (3.2)$$

That is

$$S (W_2(\xi, X)Y, Z) = \frac{1}{2n} [(\lambda + \mu)g(_\phi X, Z)\eta(Y) + (\lambda\mu + 1)(g(X, Z)\eta(Y) + \eta(x)\eta(Y)\eta(Z))] \quad (3.3)$$

and

$$S (Y, W (\xi, X)Z) = \frac{1}{2n} [(\lambda + \mu)g(_\phi X, Y)\eta(Z) + (\lambda\mu + 1)(g(X, Y)\eta(Z) + \eta(x)\eta(Y)\eta(Z))] \quad (3.4)$$

Putting $Z = \xi$, the condition (3.1) becomes

$$(\lambda + \mu)g(\phi X, Y) + (\lambda\mu + 1)g(\phi X, _\phi Y) = 0 \quad (3.5)$$

Putting $Y = {}_\phi Y$ in (3.5), we get:

$$(\lambda + \mu)g(\phi X, _\phi Y) + (\lambda\mu + 1)g(\phi X, Y) = 0. \quad (3.6)$$

Adding (3.5) and (3.6), we get:

$$(\mu + \lambda + \lambda\mu + 1)(g(\phi X, _\phi Y) + g(\phi X, Y) = 0 \quad (3.7)$$

That is

$$(\lambda + 1)(\mu + 1) = 0 \quad (3.8)$$

Thus by, the following theorem is stated.
Theorem 3.1. If $(\phi, \xi, \eta, g)$ is a Lorentzian para- Sasakian structure on the $(2n + 1)-$dimensional manifold M satisfying $(\xi,.)W^2.S = 0$ , $(g, \xi, \lambda,\mu)$ is $\eta-$Ricci soliton on M , then $\mu = -1$ and $\lambda = -2n - 1$ or $\lambda = -1$ and $\mu = 2n - 1$.
For $\mu = -_1$ we get from 3.5 that

$$(\lambda\mu + 1)g(\phi X, Y) = g(_\phi X, _\phi Y) \quad (3.9)$$

and using 3.5 in 2.18, we get

$$S(X, Y) = -2ng(X, X) \qquad (3.10)$$

Thus, we have the following theorem.
Theorem 3.2. If $(\phi, \xi, \eta, g)$ is a Lorentzian para- Sasakian structure on the $(2n + 1)-$dimensional manifold M satisfying $(\xi,.)\ W^2.S = 0$ , $(g, \xi, \lambda,\mu)$ is $\eta-$Ricci soliton on M , then for $\mu = -1$ M is Einstein
For $\lambda = -1$ we get from 3.5 that

$$g(_{\phi}X, Y) = g(_{\phi}X, {}_{\phi}Y) \qquad (3.11)$$

and from 3.5 and 2.18, we get
$S(X, Y) = -2n\eta(X)\eta(Y)$ (3.12)
Thus, we have the following theorem.
Theorem 3.3. If $(\phi, \xi, \eta, g)$ is a Lorentzian para- Sasakian structure on the $(2n + 1)-$dimensional manifold M satisfying $(\xi,.)\ W^2.S = 0$, $(g, \xi, \lambda,\mu)$ is $\eta-$Ricci soliton on M , then for $\lambda = -1$ M is Quasi- Einstein.
The following corollary is deduced from the above two theorems.
Corollary 3.4. On Lorentzian Para- Sasakian manifold $(M, \phi, \xi, \eta, g)$ satisfying $(\xi,.)\ W^2.S = 0$, there is no Ricci soliton with potential vector $\xi$.

## 4 $\eta-$Ricci Solitons on Lorentzian Para-Sasakian Manifolds satisfying $(\xi,.)\ S.W^2 = 0$.

The condition to be satisfied by S is:

$$\begin{aligned} &S(X, W_2(Y, Z)V)\xi - S(\xi, W_2(Y, Z)V)X \\ &+S(X, Y)W_2(\xi, Z)V - S(\xi, Y)W_2(X, Z)V \\ &+S(X, Z)W_2(Y, Z)V - S(\xi, Z)W_2(Y, X)V \\ &+S(X, V)W_2(Y, Z)\xi - S(\xi, V)W_2(Y, Z)X = 0. \end{aligned} \qquad (4.1)$$

Taking inner product with $\xi$ the relation (4.1) becomes:

$$\begin{aligned} &-S(X, W_2(Y, Z)V) - S(\xi, W_2(Y, Z)V)\eta(X)+ \\ &S(X, Y)\eta(W_2(\xi, Z)V) - S(\xi, Y)\eta(W_2(X, Z)V) \\ &+S(X, Z)\eta(W_2(Y, \xi)V) - S(\xi, Z)\eta(W_2(Y, X)V) \\ &+S(X, V)\eta(W_2(Y, Z)V) - S(\xi, V)\eta(W_2(Y, Z)X) = 0 \end{aligned} \qquad (4.2)$$

After using (2.18) and expanding the terms in (4.2) and putting $V = Z = \xi$ the condition to be satisfied by S becomes:

$$(\lambda + \mu)g(\phi X, Y) + (\lambda\mu + 1) = 0 \qquad (4.3)$$

Putting $Y = {}_{\phi}Y$ , we get

$$(\lambda + \mu)g(\phi X, {}_{\phi}Y) + (\lambda\mu + 1)g(\phi X, Y) = 0 \qquad (4.4)$$

After adding (4.3) and (4.4), we get

$$(\lambda + \lambda\mu + \mu + 1)[g(_{\phi}X, {}_{\phi}Y) + g(\phi X, Y)] = 0 \qquad (4.5)$$

Thus, the following theorem is stated.
Theorem 4.1. If $(\phi, \xi, \eta, g)$ is a Lorentzian para- Sasakian structure on the $(2n + 1)-$dimensional manifold M satisfying $(\xi,.)S.W_2 = 0$ , $(g, \xi, \lambda,\mu)$ is $\eta-$Ricci soliton on M , then $\mu = -1$ and $\lambda = -2n - 1$ or $\lambda = -1$ and $\mu = 2n - 1$.
For $\mu = -1$ we get from 4.3 that

$$(\lambda\mu + 1)g(\phi X, Y) = g(_{\phi}X, {}_{\phi}Y) \qquad (4.6)$$

and using 4.3 in 2.18, we get

$$S(X, Y) = -2ng(X, X) \qquad (4.7)$$

Thus, we have the following theorem.
Theorem 4.2. If $(\phi, \xi, \eta, g)$ is a Lorentzian para- Sasakian structure on the $(2n + 1)-$dimensional manifold M satisfying $(\xi,.)S.W_2 = 0$ , $(g, \xi, \lambda,\mu)$ is $\eta-$Ricci soliton on M , then for $\mu = -1$ M is Einstein
For $\lambda = -1$ we get from 4.3 that

$$g(\phi X, Y) = g(\phi X, {}_{\phi}Y) \qquad (4.8)$$

and from 4.3 and 2.18, we get

$$S(X, Y) = -2n\eta(X)\eta(Y) \quad (4.9)$$

Thus, we have the following theorem.
Theorem 4.3. If $(\phi, \xi, \eta, g)$ is a Lorentzian para- Sasakian structure on the $(2n + 1)-$dimensional manifold M satisfying $(\xi,.)S.W_2 = 0$, $(g, \xi, \lambda,\mu)$ is $\eta-$Ricci soliton on M , then for $\lambda = -1$ M is Quasi- Einstein

From the above two theorems the following corollary is deduced.
Corollary 4.4. On Lorentzian Para- Sasakian manifold $(M, \phi, \xi, \eta, g)$ satisfying $(\xi,.)$ S.W8 = 0, there is no Ricci soliton with potential vector $\xi$.

**4. Conclusion**

In this paper, we have proved that Lorentzian Para- Sasakian manifolds satisfying $(\xi,.)$ $s.W_2 = 0$ and having $\eta-$ Ricci soliton structure are Einstein or quasi-Einstein manifolds according to the value $\mu$ and $\lambda$. The same results have been proved on Manifolds satisfying $(\xi,.)$ $W_2.S = 0$. in the future we are planning to consider the same conditions on Sasakian Manifolds and on Contact manifolds. We hope to get the interesting results.

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