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Analysis of Youden square design with several missing observations

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Abstract
The need for estimating missing values has long been recognized by workers dealing with experimental design. All the numerous methods that have been advanced by various workers in this field during the last five decades can broadly be classified into three categories: non iterative procedure, iterative procedure, and covariance technique. In the present study, the authors have tried to discuss the analysis of Youden Square Design in presence of several missing observations occurring in any manner whatsoever and singular pattern of missing observations as well by using Bartlett's covariate technique. The expressions for estimates of missing observations and variance of the various elementary treatment contrasts have been obtained.

Keywords: Average variance, adjusted treatment total, Bartlett's covariate analysis, bias

Introduction
In a Youden Square Design, n = vr experimental units are arranged in v rows, r columns and v-treatments are allocated at random to these experimental units subject to the condition that each treatment occurs once in each column and each pair of treatments occurs together in λ rows. A necessary and sufficient condition for this is that a B. I. B. Design with parameters v, b = v, r, k = r, and λ exist.
The case of one missing observation was discussed by Kshirsagar and Mckee (1982) [7]. Kaushik A. K. (1992) [8] pointed out that the estimate of the missing observation and variances of the various elementary treatment contrast obtained by them seem to be incorrect. Kaushik A. K. (2010) [9, 10] discussed the case of one missing observation in a Youden Square Design in details. Later on, Kaushik A. K. and Shiv Kumar (2010) [9, 10], Kaushik A. K. and Ram Kishan (2011) [12], Kaushik A. K. (2012) [13], and Kaushik A. K. and Shiv Kumar (2019) [14] discussed the case of two missing observations in Youden Square Design in some special cases.

Material and Methods
This includes two sections. In section 1, the covariate analysis with 'p' concomitant variables is presented in brief. The subject matter discussed in this section is not entirely new but its presentation is new. It provides the relevant information and forms the basis of the present study. Section 2 deals with the subject matter under study.

Section 1
Covariate Analysis: The ANCOVA Model with 'p' concomitant variables $X_1, X_2, \ldots, X_p$ is given below

$$Y_u = \mu + \gamma_k + \delta_j + \cdots + \rho_m + t_i + X_{1u}\beta_1 + X_{2u}\beta_2 + \cdots + X_{pu}\beta_p + e_u \quad (1.1a)$$

Its corresponding matrix model is

$$Y = Z\pi + At + X_1\beta_1 + X_2\beta_2 + \cdots + X_p\beta_p + e \quad (1.1b)$$

with usual standard notations. The error sum of square will be

$$\text{E.S.S.} = \text{min. of } (Y - Z\pi - At - X_1\beta_1 - X_2\beta_2 - \cdots - X_p\beta_p)' (Y - Z\pi - At - X_1\beta_1 - X_2\beta_2 - \cdots - X_p\beta_p) \quad (1.2)$$

with respect to $\pi, t, \beta_1, \beta_2, \ldots,$ and β_p only. The normal equations are:

$$Z'Z\hat{\pi} + Z'A\hat{t} + Z'X_1\hat{\beta}_1 + Z'X_2\hat{\beta}_2 + \cdots + Z'X_p\hat{\beta}_p = Z'Y \quad (1.3)$$
$$A'Z\hat{\pi} + A'A\hat{t} + A'X_1\hat{\beta}_1 + A'X_2\hat{\beta}_2 + \cdots + A'X_p\hat{\beta}_p = A'Y \quad (1.4)$$
$$X_1'Z\hat{\pi} + X_1'A\hat{t} + X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 + \cdots + X_1'X_p\hat{\beta}_p = X_1'Y \quad (1.5.1)$$
$$X_2'Z\hat{\pi} + X_2'A\hat{t} + X_2'X_1\hat{\beta}_1 + X_2'X_2\hat{\beta}_2 + \cdots + X_2'X_p\hat{\beta}_p = X_2'Y \quad (1.5.2)$$
$$X_p'Z\hat{\pi} + X_p'A\hat{t} + X_p'X_1\hat{\beta}_1 + X_p'X_2\hat{\beta}_2 + \cdots + X_p'X_p\hat{\beta}_p = X_p'Y \quad (1.5.p)$$

Solving the above normal equations, we can get

$$\hat{\pi} = (Z'Z)^{-1}(Z'Y - Z'A\hat{t} - Z'X_1\hat{\beta}_1 - Z'X_2\hat{\beta}_2 - \cdots - Z'X_p\hat{\beta}_p) \quad (1.6.1)$$
$$\hat{t} = \bar{C}(Q_{(y)} - Q_{(X_1)}\hat{\beta}_1 - Q_{(X_2)}\hat{\beta}_2 - \cdots - Q_{(X_p)}\hat{\beta}_p) \quad (1.6.2)$$

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\beta}_p \end{bmatrix} = \emptyset^{-1} \begin{bmatrix} E_{X_1Y} \\ E_{X_2Y} \\ \cdot \\ \cdot \\ \cdot \\ E_{pY} \end{bmatrix} \quad (1.6.3a)$$

Where

$$\emptyset = \begin{bmatrix} E_{X_1X_1} & E_{X_1X_2} & \cdots & E_{X_1X_p} \\ E_{X_2X_1} & E_{X_2X_2} & \cdots & E_{X_2X_p} \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ E_{X_pX_1} & E_{X_pX_2} & \cdots & E_{X_pX_p} \end{bmatrix} \quad (1.6.3b)$$

The error sum of square is

$$\text{E.S.S.} = Y'Y - Y'Z\hat{\pi} - Y'A\hat{t} - Y'X_1\hat{\beta}_1 - Y'X_2\hat{\beta}_2 - \cdots - Y'X_p\hat{\beta}_p$$

$$= E_{YY} - \left[E_{X_1Y}\, E_{X_2Y} \ldots E_{X_pY}\right] \emptyset^{-1} \begin{bmatrix} E_{X_1Y} \\ E_{X_2Y} \\ \cdot \\ \cdot \\ \cdot \\ E_{X_pY} \end{bmatrix} \quad (1.7)$$

with (ν – p) df only.
Under the null hypothesis
H_0: $t_1 = t_2 = \ldots = t_v = 0$
The model (1.1a) reduces to

$$Y_u = \mu + \gamma_k + \delta_j + \cdots + \rho_m + X_{1u}\beta_1 + X_{2u}\beta_2 + \cdots + X_{pu}\beta_p + e_u \quad (1.8a)$$

Its corresponding matrix model is

$$Y = Z\pi + X_1\beta_1 + X_2\beta_2 + \cdots + X_p\beta_p + e \quad (1.8b)$$
$$E_0.\text{ S. S.} = \text{min. of } (Y - Z\pi - X_1\beta_1 - X_2\beta_2 - \cdots - X_p\beta_p)'$$
$$(Y - Z\pi - X_1\beta_1 - X_2\beta_2 - \cdots - X_p\beta_p) \quad (1.9)$$

with respect to $\pi, \beta_1, \beta_2, \ldots,$ and β_p only. The normal equations are:

$$Z'Z\pi^* + Z'X_1\beta_1^* + Z'X_2\beta_2^* + \cdots + Z'X_p\beta_p^* = Z'Y \quad (1.10)$$
$$X_1'Z\pi^* + X_1'X_1\beta_1^* + X_1'X_2\beta_2^* + \cdots + X_1'X_p\beta_p^* = X_1'Y \quad (1.11.1)$$
$$X_2'Z\pi^* + X_2'X_1\beta_1^* + X_2'X_2\beta_2^* + \cdots + X_2'X_p\beta_p^* = X_2'Y \quad (1.11.2)$$
$$X_p'Z\pi^* + X_p'X_1\beta_1^* + X_p'X_2\beta_2^* + \cdots + X_p'X_p\beta_p^* = X_p'Y \quad (1.11.p)$$

Solving these equations, we can get

$$\pi^* = (Z'Z)^{-1}(Z'Y - Z'X_1\beta_1^* - Z'X_2\beta_2^* - \cdots - Z'X_p\beta_p^*) \quad (1.12)$$

$$\begin{bmatrix} \beta_1^* \\ \beta_2^* \\ \cdot \\ \cdot \\ \cdot \\ \beta_p^* \end{bmatrix} = \emptyset^{*-1} \begin{bmatrix} E_{X_1Y}^* \\ E_{X_2Y}^* \\ \cdot \\ \cdot \\ \cdot \\ E_{X_pY}^* \end{bmatrix} \quad (1.13a)$$

Where

$$\emptyset^* = \begin{bmatrix} E_{X_1X_1}^* \; E_{X_1X_2}^* \ldots E_{X_1X_p}^* \\ E_{X_2X_1}^* \; E_{X_2X_2}^* \ldots E_{X_2X_p}^* \\ \cdot \\ \cdot \\ \cdot \\ E_{X_pX_1}^* \; E_{X_pX_2}^* \ldots E_{X_pX_p}^* \end{bmatrix} \quad (1.13b)$$

The error sum of square will be

E$_0$. S. S. = $Y'Y - Y'Z\pi^* - Y'X_1\beta_1^* - Y'X_2\beta_2^* - \cdots - Y'X_p\beta_p^*$

$$= E_{YY}^* - \left[E_{X_1Y}^* \; E_{X_2Y}^* \ldots E_{X_pY}^*\right] \emptyset^{*-1} \begin{bmatrix} E_{X_1Y}^* \\ E_{X_2Y}^* \\ \cdot \\ \cdot \\ \cdot \\ E_{X_pY}^* \end{bmatrix} \quad (1.14)$$

with (v + v – p –1) df only.
Treatment S. S. = E$_0$. S. S. − E. S. S. (1.15)
with (v −1) df only.
From eq^n (1.6.2), we get
$V(\hat{t}) = \bar{C}\, V\{Q_{(Y)}\}\, \bar{C} + \bar{C}\, Q\emptyset^{-1}Q'\bar{C}\sigma^2$
$= \bar{C}C\, \sigma^2\bar{C} + \bar{C}\, Q\emptyset^{-1}Q'\bar{C}\sigma^2$
$= \bar{C}\, \sigma^2 + M\, \emptyset^{-1}M'\sigma^2$ (1.16)

Where

$$Q' = \begin{bmatrix} Q_{(X_1)} \\ Q_{(X_2)} \\ \cdot \\ \cdot \\ \cdot \\ Q_{(X_p)} \end{bmatrix} = \begin{bmatrix} Q_{1(X_1)} \; Q_{2(X_1)} \ldots Q_{p(X_1)} \\ Q_{1(X_2)} \; Q_{2(X_2)} \ldots Q_{p(X_p)} \\ \cdot \\ \cdot \\ \cdot \\ Q_{1(X_p)} \; Q_{2(X_2)} \ldots Q_{p(X_p)} \end{bmatrix} \quad (1.16a)$$

$$M' = \begin{bmatrix} \hat{t}_{1(X_1)} \; \hat{t}_{2(X_1)} \ldots \hat{t}_{v(X_1)} \\ \hat{t}_{1(X_2)} \; \hat{t}_{2(X_2)} \ldots \hat{t}_{v(X_2)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{t}_{1(X_p)} \; \hat{t}_{2(X_p)} \ldots \hat{t}_{v(X_p)} \end{bmatrix} \quad (1.16b)$$

Average Variance = $2a\,\sigma^2 + \frac{2}{(v-1)}\, tr. M\, \emptyset^{-1}\, M'\sigma^2$ (1.17)

Further discussion on this topic is not relevant to the present study and hence not been discussed.

Section 2

Let the experiment be conducted in a Youden Square obtained from a symmetric BIBD (v = b, r = k, λ). Suppose 'm' cell observations are missing in any manner whatsoever in the experiment. Without any loss of generality, we may assume that these are the first 'm' observations Y_1, Y_2, …, Y_m respectively. The author has described the analysis of one and two missing observations in all possible cases in a Youden Square in his previous research papers. The generalization of these leads to the following result.

$$\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{Y}_m \end{bmatrix} = \frac{1}{\lambda vk} \emptyset^{-1} \begin{bmatrix} \lambda v R_{(1)} + \lambda k C_{(1)} + k\{kQ_{(1)} - Q'_{(1)}\} - \lambda G \\ \lambda v R_{(2)} + \lambda k C_{(2)} + k\{kQ_{(2)} - Q'_{(2)}\} - \lambda G \\ \cdot \\ \cdot \\ \cdot \\ \lambda v R_{(m)} + \lambda k C_{(m)} + k\{kQ_{(m)} - Q'_{(m)}\} - \lambda G \end{bmatrix} \qquad (2.1)$$

Where

$\emptyset = (\emptyset_{pp})$ is a matrix of order 'm' with

$\emptyset_{pp} = \frac{k(k-1)(k-2)}{\lambda vk}$, p = 1, 2, …, m

$\emptyset_{pp'} = -\frac{k(k-2)}{\lambda vk}$, if p - th and p'- th missing observation belong to the same

row or same treatment.

$= -\frac{k(\lambda-2)}{\lambda vk}$, if p - th and p'- th missing observation belong to different rows, same column, and different treatments which are common in both the rows.

$= -\frac{k(\lambda-1)}{\lambda vk}$, if p - th and p'- th missing observation belong to different rows, same column, and different treatments one treatment being common in both the rows while other is not.

$= -\frac{\lambda k}{\lambda vk}$, if p - th and p'- th missing observation belong to different rows, same column, and different treatments which are not common in both the rows.

$= \frac{k}{\lambda vk}$, if p - th and p'- th missing observation belong to different rows, different columns, and different treatments one treatment being common in both the rows while other is not.

= 0, if p - th and p'- th missing observation belong to different rows, different columns, and different treatments which are not common in both the rows.

$= \frac{2k}{\lambda vk}$, if p - th and p'- th missing observation belong to different Rows, different columns, and different treatments which are common in both the rows.

$R_{(p)}$ = total of all the known cell observations of the row to which p – th missing observation belongs to,

$C_{(p)}$ = total of all the known cell observations of the column to which p – th missing observation belongs to,

$Q_{(p)}$ = adjusted treatment total of the treatment to which p - th missing observation belongs to,

$Q'_{(p)}$ = total of all the adjusted treatment totals of the block to which p - th missing observation belongs to, and

G = total of all the known cell observations in the experiment.

The expression for the $V(\hat{t}\,)$ can be obtained as below

$$V(\hat{t}\,) = \frac{k\sigma^2}{\lambda v} I_v + M\emptyset^{-1}M'\sigma^2 \qquad (2.2)$$

Where

$M = (\hat{t}_{ip})_{v\times m}$

$\hat{t}_{ip} = \frac{k}{\lambda v}\left(1 - \frac{1}{k}\right)$, if p - th missing observation belongs to i – th treatment,

$= -\frac{k}{\lambda vk}$, if the treatment is present in the block which contains the missing observation but it does not belong to it, and

= 0, otherwise.

$$\text{Average Variance} = \frac{2k\sigma^2}{\lambda v} + \frac{2}{(v-1)} tr. M\emptyset^{-1}M'\sigma^2 \qquad (2.3)$$

Special Case 1: The Missing Observations Belong to the Same Row (m < k)

Without any loss of generality, we may assume that all the 'm' (m < k) missing observations belong to the first row so that the estimates of the missing observations will be obtained by

$$\hat{Y}_p = \frac{(k-m)\varphi_p + \varphi'_p}{k^2(k-m)(k-2)}, p = 1, 2, \dots, m \qquad (2.4)$$

Where

$$\varphi_p = \lambda v R_{(1)} + \lambda k C_{(p)} + k\{kQ_{(p)} - Q'_{(p)}\} - \lambda G$$

$$\varphi'_{(p)} = \sum_{p=1}^{m} \varphi_{(p)}$$

Under the null hypothesis

H_0: $t_1 = t_2 = \dots = t_v = 0$

The new estimates of the missing observations will be

$$Y_p^* = \frac{(k-m)Q_{(p)}^* + \sum_{p=1}^{m} Q_{(p)}^*}{k(k-m)(v-1)} \quad (2.5)$$

The variance covariance matrix will be

$$V(\hat{t}\,) = \frac{k\sigma^2}{\lambda v} I_v + M\emptyset^{-1}M'\sigma^2 \quad (2.6)$$

Where

$$\emptyset^{-1} = \frac{\lambda v}{k(k-m)(k-2)} \{(k-m)I_m + E_{m\times m}\},$$

I_m is the unit matrix of order 'm' and $E_{m\times m}$ is a square matrix having each element as unity, and

$$M' = \frac{1}{\lambda v}\begin{bmatrix} k-1 & -1 & \dots & -1 & -1 & \dots & -1\,0 & \dots & 0 \\ -1 & k-1 & \dots & -1 & -1 & \dots & -1\,0 & \dots & 0 \\ & & & \vdots & & & & & \\ & & & \vdots & & & & & \\ -1 & -1 & \dots & k-1 & -1 & \dots & -1\,0 & \dots & 0 \end{bmatrix}$$

$$V(\hat{t}_p - \hat{t}_{p'}) = \frac{2k}{\lambda v}\sigma^2 + \frac{2k}{\lambda v(k-2)}\sigma^2 \quad (2.7)$$

$p \neq p' = 1,2,\dots,m$

$$V(\hat{t}_p - \hat{t}_u) = \frac{2k}{\lambda v}\sigma^2 + \frac{k(k-m+1)}{\lambda v(k-m)(k-2)}\sigma^2 \quad (2.8)$$

$u = m+1, m+2, \dots, k$

$$V(\hat{t}_p - \hat{t}_w) = \frac{2k}{\lambda v}\sigma^2 + \frac{(k-1)}{\lambda v(k-2)}\sigma^2 \quad (2.9)$$

$w = k+1, k+2, \dots, v$

$$V(\hat{t}_u - \hat{t}_w) = \frac{2k}{\lambda v}\sigma^2 + \frac{m}{\lambda v(k-2)(k-m)}\sigma^2 \quad (2.10)$$

$$V(\hat{t}_u - \hat{t}_{u'}) = V(\hat{t}_w - \hat{t}_{w'}) = \frac{2k}{\lambda v}\sigma^2 \quad (2.11)$$

$u \neq u'\ w \neq w'$

Average Variance = $\frac{2k}{\lambda v}\sigma^2 + \frac{2mk}{\lambda v(v-1)(k-2)}\sigma^2$ (2.12)

Relative Efficiency = $\frac{(v-1)(k-2)}{(v-1)(v-2)+m}$ (2.13)

Relative Loss in Efficiency = $\frac{m}{(v-1)(v-2)+m}$ (2.14)

Bias = $\frac{(v-1)(k-1)}{vk}\sum_{p=1}^{m}(\hat{Y}_p - Y_p^*)^2 - \frac{2(v-1)}{vk}\sum\sum(\hat{Y}_p - Y_p^*)(\hat{Y}_{p'} - Y_{p'}^*)$ (2.15)

$p \neq p'$

Special Case 2: Singular Pattern of Missing Observations (A Row is Entirely Missing)
Without any loss of generality, we may assume that the first row has been entirely missing from the Youden Square so that the estimates of the missing observations will be obtained by

$$\frac{k(k-2)}{\lambda vk}\begin{bmatrix} k-1 & -1 & \dots & -1 \\ -1 & k-1 & \dots & -1 \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ -1 & -1 & \dots & k-1 \end{bmatrix}\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{Y}_k \end{bmatrix} = \frac{1}{\lambda vk}\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_P \end{bmatrix} \quad (2.16)\ Where\ \varphi_p = \lambda k C_{(p)} + k\{kQ_{(p)} - Q'_{(P)}\} - \lambda G$$

which represents a dependent set of linear equations. Hence, we impose the restriction $\sum_{p=1}^{k} \hat{Y}_p = \frac{G}{(v-1)}$ (2.17)

Under the above restriction, eq^n (2.16) reduces to

$$\frac{k^2(k-2)}{\lambda vk}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & . & \\ & & . & \\ & & . & \\ 0 & 0 & \dots & 1 \end{bmatrix}\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ . \\ . \\ . \\ \hat{Y}_k \end{bmatrix} = \frac{1}{\lambda vk}\begin{bmatrix} \varphi_1^* \\ \varphi_2^* \\ . \\ . \\ . \\ \varphi_k^* \end{bmatrix} \quad (2.18)$$

Where $\varphi_p^* = \varphi_p + \frac{k(k-2)G}{v-1}$ and $\hat{Y}_p = \frac{1}{k(k-2)}\left[\lambda C_{(p)} + \left\{kQ_{(p)} - Q'_{(P)}\right\} - \frac{G}{(v-1)}\right]$ (2.19)

p = 1, 2, …, k

The variance covariance matrix will be

$$V(\hat{t}\,) = \frac{k\sigma^2}{\lambda v} I_v + M\overline{\emptyset}M'\sigma^2 \quad (2.20)$$

Where $\overline{\emptyset} = \frac{\lambda vk}{k^2(k-2)} I_v$

$$M' = \frac{1}{\lambda v}\begin{bmatrix} k-1 & -1 & \dots & -1 & 0 & \dots & 0 \\ -1 & k-1 & \dots & -1 & 0 & \dots & 0 \\ & & & . & & & \\ & & & . & & & \\ & & & . & & & \\ -1 & -1 & \dots & k-1 & 0 & \dots & 0 \end{bmatrix}$$

$$V\left(\hat{t}_p - \hat{t}_{p'}\right) = \frac{2k\sigma^2}{\lambda v} + \frac{2k\sigma^2}{\lambda v(k-2)} \quad (2.21)$$

$p \neq p' = 1, 2, \dots, k$

$$V\left(\hat{t}_p - \hat{t}_w\right) = \frac{2k\sigma^2}{\lambda v} + \frac{(k-1)\sigma^2}{\lambda v(k-2)} \quad (2.22)$$

$w = k+1, k+2, \dots, v$

$$V(\hat{t}_w - \hat{t}_{w'}) = \frac{2k\sigma^2}{\lambda v} \quad (2.23)$$

$w \neq w'$

Average Variance = $\frac{2k\sigma^2}{\lambda v} + \frac{2k(k-1)\sigma^2}{\lambda v(v-1)(k-2)}$ (2.24)

Relative Efficiency = $\frac{(v-1)(k-2)}{v(k-2)+1}$ (2.25)

Relative Loss in Efficiency = $\frac{(k-1)}{v(k-2)+1}$ (2.26)

Bias = $\frac{(v-1)}{v} \sum_{p=1}^{k}(\bar{C}_p - \hat{Y}_p)^2$ (2.27)

Where $\bar{C}_p = \frac{C_p}{(v-1)}$

Example: Estimate the missing observations and analyse the data:

Rows	Columns			
	I	II	III	IV
1	A = –	B = 9	C = 0	D = 14
2	B = 6	A = 5	E = 5	C = 19
3	C = 1	D = 9	A = 0	E = 7
4	D = 8	E = 8	B = –	A = –
5	E = 7	C = 6	D = 11	B = 10

Solution: Let us assume that the first missing observation belong to treatment A in row I, second to treatment B in row IV, and third to treatment A in row IV respectively. These are denoted by Y_1, Y_2, and Y_3 respectively.
The estimates of the missing observations are

$$\begin{bmatrix}\hat{Y}_1\\ \hat{Y}_2\\ \hat{Y}_3\end{bmatrix} = \begin{bmatrix}24 & 8 & -8\\ 8 & 24 & -8\\ -8 & 8 & 24\end{bmatrix}^{-1}\begin{bmatrix}84\\ 84\\ 48\end{bmatrix} = \begin{bmatrix}3.75\\ 3.75\\ 4.5\end{bmatrix}$$

E. S. S. $= \{(3.75)^2 + (3.75)^2 + (4.50)^2 + \sum Y_a^2\} - \frac{1}{4}\{(R_1 + 3.75)^2 + R_2^2 + R_3^2 + (R_4 + 3.75 + 4.50)^2 + R_5^2\} - \frac{1}{5}\{(C_1 + 3.75)^2 + C_2^2 + (C_3 + 3.75)^2 + (C_4 + 4.50)^2\} - \frac{4}{5}\sum Q_i^2 + \frac{(G+3.75+3.75+4.50)^2}{20} = 8.4333\ with\ 5\ df\ only.$

Under the null hypothesis

H_0: $t_A = t_B = t_C = t_D = t_E = 0$

We get the new estimates of the missing observations as

$$\begin{bmatrix}Y_1^*\\ Y_2^*\\ Y_3^*\end{bmatrix} = \begin{bmatrix}12 & 1 & 1\\ 1 & 12 & -4\\ 1 & -4 & 12\end{bmatrix}^{-1}\begin{bmatrix}94\\ 35\\ 107\end{bmatrix} = \begin{bmatrix}6.489\\ 5.814\\ 10.314\end{bmatrix}$$

E_0. S. S. $= \{(6.489)^2 + (5.814)^2 + (10.314)^2 + \sum Y_a^2\} - \frac{1}{4}\{(R_1 + 6.489)^2 + R_2^2 + R_3^2 + (R_4 + 10.314)^2 + R_5^2\} - \frac{1}{5}\{(C_1 + 6.489)^2 + C_2^2 + (C_3 + 5.814)^2 + (C_4 + 10.314)^2\} + \frac{(G+6.489+5.814+10.314)^2}{20} = 134.4467\ with\ 9\ df\ only.$

Treatment S. S. = E_0. S. S. – E. S. S. = 126.0134 with 4 df only.

ANOVA Table

Source of Variation	d.f.	S.S.	M. S.	Variance Ratio	$F_{0.05}$
Treatment	4	126.0134	31.5034	18.68	6.35
Error	5	8.4333	1.6867	–	–
Total	9	134.4467	–	–	–

The variance covariance matrix will be

$$V(\hat{t}\,) = \frac{4\sigma^2}{15}I_5 + \frac{1}{600}\begin{bmatrix}3 & -1 & 3\\ -1 & 3 & -1\\ -1 & 0 & 0\\ -1 & -1 & -1\\ 0 & -1 & -1\end{bmatrix}\begin{bmatrix}24 & 8 & -8\\ 8 & 24 & -8\\ -8 & 8 & 24\end{bmatrix}^{-1}\begin{bmatrix}3 & -1 & -1 & -1 & 0\\ -1 & 3 & 0 & -1 & -1\\ 3 & -1 & 0 & -1 & -1\end{bmatrix}$$

$$V(\hat{t}_1 - \hat{t}_2) = \frac{8\sigma^2}{15} + \frac{56\sigma^2}{75}, V(\hat{t}_1 - \hat{t}_3) = \frac{8\sigma^2}{15} + \frac{13\sigma^2}{30},$$

$$V(\hat{t}_1 - \hat{t}_4) = \frac{8\sigma^2}{15} + \frac{8\sigma^2}{15}, V(\hat{t}_1 - \hat{t}_5) = \frac{8\sigma^2}{15} + \frac{31\sigma^2}{75},$$

$$V(\hat{t}_2 - \hat{t}_3) = \frac{8\sigma^2}{15} + \frac{17\sigma^2}{150}, V(\hat{t}_2 - \hat{t}_4) = \frac{8\sigma^2}{15} + \frac{16\sigma^2}{75},$$

$$V(\hat{t}_2 - \hat{t}_5) = \frac{8\sigma^2}{15} + \frac{19\sigma^2}{75}, V(\hat{t}_3 - \hat{t}_5) = \frac{8\sigma^2}{15} + \frac{7\sigma^2}{150},$$

$$V(\hat{t}_3 - \hat{t}_4) = \frac{8\sigma^2}{15} + \frac{\sigma^2}{30}, V(\hat{t}_2 - \hat{t}_5) = \frac{8\sigma^2}{15} + \frac{\sigma^2}{75},$$

Average Variance = $\frac{2}{(v-1)}\left(tr. - \frac{\Delta}{v}\right) = \frac{8\sigma^2}{15} + \frac{7\sigma^2}{25}$

Relative Efficiency = $\frac{40}{61}$

Relative Loss in Efficiency = $\left(1 - \frac{40}{61}\right) \times 100 = 34.426\%$

Conclusion: F = 18.68 > 6.35, hence null hypothesis is rejected at 5% level of significance and we conclude that the treatments are not homogeneous.

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