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η- ricci solitons defined with W_8− curvature tensor and cyclic ricci tensor on para-kenmotsu manifolds

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Abstract
In this Paper η- Ricci solitons are considered on Para- Kenmotsu manifolds satisfying $(\xi,.)_S.W_8 = 0$ and $(\xi,.)_{W8}.S = 0$. The results of Blaga [1] for W_2 have motivated us to use the same conditions on W_8. We have proved that the Para- Kenmotsu manifolds satisfying $(\xi,.)_{W8}.S = 0$. Are quasi- Einstein Manifolds and those satisfying $(\xi,.)_S.W_8 = 0$, are Einstein Manifolds. At the end of the paper it has been proven that the para- Kenmotsu manifolds with cyclic Ricci tensor and $\eta-$ Ricci soliton structure are quasi-Einstein manifolds.

Keywords: Ricci solitons, $\eta-$ricci solitons, para-kenmotsu manifolds

1. Introduction

The Poincare Conjecture has been one of the entente century problem which have taken long time before being proved by Perelman. The solution of this problem has been possible due to the method introduced by Hamilton [2] in 1982. From this time to now different authors have studied Ricci solitons on various manifolds. Among them are Chow and Bennett [3] who considered Ricci flow on the 2-sphere and showed that the Gaussian curvature of any metric on S^2 becomes positive in finite time. Chow, Bennett [3] and Chu, Sun-Chin and Glickenstein [4] used Maximum principle to control various geometric quantities associated to the metric under Ricci flow and Chandra, Soumen and Hui [5] used Second order parallel tensors to find the conditions of Ricci solitons on Lorentzian concircular structure n-manifolds to be shrinking, steady and expending. It is noted that Ricci soliton are the solutions of Ricci flows, which move only by one parameter group of diffeomorphism and scaling, that is a Ricci soliton (g,v,λ) on Reimannian manifold (M,g) is generalization of Einstein metric such that.

$$L_v g + 2S + 2\lambda g = 0. \tag{1.1}$$

Where L_V is the Lie derivative along the vector V on M, S is Ricci tensor, λ is a scalar and g is Reimannian metric on M.
After Perelman [6] used Ricci flow and its surgery to prove the Poincare Conjecture, most mathematicians have been interested in the study of Ricci solitons. Huisken [7] was the first to study the Ricci flows on a manifold of dimensions greater than four basing his analysis on the decomposition of the Riemann curvature tensor as follows.

$$Rijkl = Uijkl + Vijkl + Wijkl,$$

Where U_{ijkl} is curvature tensor associated with the scalar curvature, V_{ijkl} is curvature tensor associated with trace free curvature and W_{ijkl} is Weyl tensor. From this time more mathematicians have investigated the properties of Ricci flows solutions especially the existence of Ricci solitons in some particular directions under certain conditions. Due to the results obtained by Huisken and noting that Pokhariyal [8] has defined W_8 curvature tensor with help of Weyl's projective tensor, we have classified Ricci solitons on para-Kenmotsu manifolds satisfying some conditions with respect to W_8 in direction of Characteristic vector ξ.
Blaga [1] have studied $\eta-$Ricci solitons on para-Kenmotsu geometry manifolds satisfying $(\xi,\cdot)_R\cdot$

$(\xi,\cdot)_R \cdot S = 0, (\xi,\cdot)_S \cdot R = 0, (\xi,\cdot)_{W_2} \cdot S = 0$ and $(\xi,\cdot)_S \cdot W_2 = 0$. Also Nagararaja and Premalata [9] obtained some results on Rcci solitons satisfying $(\xi,\cdot)_H \cdot S = 0, (\xi,\cdot)_{\tilde{C}} \cdot S = 0, (\xi,\cdot)_R \cdot \tilde{C} = 0$, $(\xi,\cdot)_P \cdot \tilde{C} = 0$ and Bagewadi, Ingalahalli and Ashoka [10] have considered the cases of $(\xi,\cdot)_R \cdot B = 0$, $(\xi,\cdot)_B \cdot S = 0$, $(\xi,\cdot)_S \cdot R = 0$, $(\xi,\cdot)_R \cdot \bar{P} = 0$ and $(\xi,\cdot)_{\bar{P}} \cdot S = 0$. In the present paper Almost para contact manifolds are considered and precisely η- Ricci soliton are studied on para- Kenmotsu manifold satisfying $(\xi,\cdot)_{W_8} \cdot S = 0$, $(\xi,\cdot)_S \cdot W_8 = 0$ and those with cyclic Ricci tensor.

2. Para-kenmotsu manifolds

Definition 2.1. *An* $(2n + 1)$: dimensional manifolds is almost para-contact if:

$$\phi\xi = 0 \tag{2.1}$$
$$\eta(\xi) = 1 \tag{2.2}$$
$$\phi^2(X) = X - \eta(X) \otimes \xi \tag{2.3}$$
$$g(\phi.,\phi.) = -g + \eta \otimes \eta, \tag{2.4}$$

(M,ϕ,ξ,η,g) is called almost paracontact manifold, ϕ the structure endomorphism, ξ the characteristic vector field and η the Paracontact form. Examples of almost paracontact metric structure are given in [11]. From the definition it follows that:

$$\eta(X) = g(X,\xi) \tag{2.5}$$

and ξ is a unit vector field.

Definition 2.2. An almost paracontact metric structure (M,ϕ,ξ,η,g) is said to be ParaKenmotsu if the Levi- Civita connection ∇ of g satisfies the following equation:

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X \forall X,Y \in \mathfrak{X}(M), \tag{2.6}$$

where X(M) is the algebra of vector fields on M.

Para- Kenmotsu structure has been introduced by Welyczko in [12] for 3- dimensional normal almost paracontact metric structure. In the following we give the fundamental properties properties of this structure and their proofs can be found in [1].

Proposition 2.3. On Para-Kenmotsu manifolds the followings hold:

$$\nabla\xi X = X - \eta \otimes \xi(x) \tag{2.7}$$

$$\eta(\nabla_X\xi) = 0 \tag{2.8}$$

$$\nabla_\xi\xi = 0 \tag{2.9}$$

$$R(X,Y)\xi = -\eta(X)Y - \eta(y)X \tag{2.10}$$

$$R(X,Y)Z = -Xg(Y,Z) + Y g(X,Z) \tag{2.11}$$

$$\nabla\eta = g - \eta \otimes \eta \tag{2.12}$$

$$\nabla_\xi\eta = 0 \tag{2.13}$$

$$L_\xi\phi = 0 \tag{2.14}$$

$$L_\xi\eta = 0 \tag{2.15}$$

$$L_\xi(\eta \otimes \eta) = 0 \tag{2.16}$$

$$L_\xi g = 2(g - \eta \otimes \eta) \tag{2.17}$$

$$\nabla_X\xi = X - \eta(X)\xi, \tag{2.18}$$

where R is Riemann curvature tensor field and ∇ is Levi- Civita connection of g. Example of para- Kenmotsu structure can be find in [1].

3. η−Ricci Solitons on Para- Kenmotsu manifolds

If (M,ϕ,ξ,η,g) is an almost paracontact metric manifold, (g,ξ,λ,μ) satisfying

$$L_\xi g + 2S + 3\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where L_ξ is the lie derivative along the characteristic vector field, S is the Ricci curvature tensor of g, and λ and μ are constant, then (g,ξ,λ,μ) is called $\eta-$ Ricci soliton structure on M. Note that (3.1) become equation of Ricci soliton for $\mu = 0$ and it is called shrinking, steady or expending according to λ is negative, zero or positive, respectively. In terms of Levi Civita connection (3.1) becomes:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X,\nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y). \tag{3.2}$$

Blaga in [1] has showed that one of important geometrical object to the study of Ricci Solitons is a symmetric tensor (0,2) which is parallel with respect to the Levi- Civita connection and some of geometric properties of such tensor field are defined in [13]. Considering such symmetric (0,2) tensor field α and from the Ricci identity:

$$\nabla^2\alpha(X,Y;Z,V) - \nabla^2\alpha(X,Y;V,Z) = 0, \tag{3.3}$$

we get

$$\alpha(R(X,Y)Z,V) + \alpha(Z,R(X,Y)V) = 0, \tag{3.4}$$

$\forall X,Y,Z,V \in \mathfrak{X}(M)$. Taking $Z = V = \xi$ and using (2.10), we get

$\alpha(R(X,y)\xi,\xi) = 0, \forall X,Y \in \mathfrak{X}(M)$.

Using (2.18), (3.2) becomes

$$S(X,Y) = -(\lambda + 1)g(X,Y) - (\mu - 1)\eta(X)\eta(Y). \tag{3.5}$$

Definition 3.1. A manifold is called quasi- Einstein if the Ricci curvature tensor field S is linear combination of g and the tensor product of a non- zero 1-form η satisfying $\eta(x) = g(X, \xi)$, for ξ a unit vector field, and it is Einstein if S is collinear with g. Blaga has proved in [1] the following theorem.

Theorem 3.2. Let (M, ϕ, ξ, η, g) be a para-Kenmotsu manifold. Assume that the symmetric (0, 2)- tensor field $\beta = L_\xi g+2s+2\mu\otimes\eta$ is parallel with respect to the Levi- Civita connection associated to g. Then (g,ξ,μ) yields an $\eta-$ Ricci soliton. From this theorem we can state for $\mu = 0$, the following corollary.

Corollary 3.3. On para-Kenmotsu manifold (M, ϕ, ξ, η, g) with the property that the symmetric (0,2)- tensor field $\alpha = L_\xi g + 2s$ is parallel with respect to Levi-Civita connection associated to g the relation (3.1), defines a Ricci soliton on M. for $\lambda = 2n$ and $\mu = 0$.

Corollary 3.4. For $\mu = 1$ and $\lambda = 2n - 1$, (M,g) is quasi- Einstein.
In the following we shall study η-Ricci solitons whose curvature satisfies $(\xi,.)_S.W_8 = 0$ and $(\xi,.)_{W8}.S = 0$ respectively, where the W_8-curvature tensor has been introduced by Pokhariyal [8], and it is given by:

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{2n}(S(X,Y)Z - S(Y,Z)X) \tag{3.6}$$

for a $(2n + 1)-$ dimensional Para-Kenmotsu manifold.

4. η- Ricci solitons on para- Kenmotsu manifolds satisfying $(\xi,.)W_8.S = 0$**.**
The condition to be satisfied by S is [1]:

$$S(W_8(\xi,X)Y,Z) + S(Y,W_8(\xi,X)Z) = 0. \tag{4.1}$$

$\forall X,Y,Z \in \mathfrak{X}(M)$.

Using (3.5) and (3.6) in (4.1), we get:

$$(\mu - \lambda - 2)\{(g(x,y)\eta(z) + g(X,Z)\eta(Y)) - 2g(Y,Z)\eta(x) - 2(\mu - 1)\eta(x)\eta(y)\eta(z)\} = 0 \tag{4.2}$$

$\forall X,Y,Z \in \mathfrak{X}(M)$ Making $Z = \xi$ we get

$$(\mu - \lambda - 2)(g(X,Y) - \eta(X)\eta(Y)) = 0 \tag{4.3}$$

Or

$$(\mu - \lambda - 2)g(\phi X,\phi Y) = 0 \tag{4.4}$$

$\forall X,Y \in \mathfrak{X}(M)$

But $\lambda + \mu = 2n$, so $\mu - (2n - \mu) - 2 = 0$, or $2\mu = 2n + 2$. Hence, we have the following theorem.

Theorem 4.1. If (ϕ,ξ,η,g) is a para- Kenmotsu structure on the $(2n + 1)$- dimensional manifold M, (g,ξ,η,λ,μ) is an η-Ricci soliton on M satisfying $(\xi,.)_{W8}.S = 0$, then $\mu = n+1$ and $\lambda = n - 1$.

Corollary 4.2. If (ϕ,ξ,η,g) is a para- Kenmotsu structure on the $(2n + 1)$- dimensional manifold M, (g,ξ,η,λ,μ) is an η-Ricci soliton on M satisfying $(\xi,.)_{W8}.S = 0$, then M is quasi- Einstein.
Proof. As μ 6= 0 and by using definition (3.1), the Corollary is deduced from the above theorem by using the expression 3.1.

Corollary 4.3. On a para-Kenmotsu manifold (M,ϕ,ξ,η,g)satisfying $(\xi,.)_{W8}.S = 0$, there is no Ricci solitons with potential vector field ξ.
Proof. The relation (3.1) is Ricci soliton if μ = 0 and λ 6= 0 but μ = 0 implies n = −1,that is λ = −2 which is a contradiction as dimension of manifold is always positive and $\mu+\lambda = 2n$.

5. *η*- Ricci solitons on para-Kenmotsu manifolds satisfying

$(\xi,.)_S.W_8 = 0$.

The condition to be satisfied by *S* [1] is:
$S(X,W_8(Y,Z)V)\xi - S(\xi,W_8(Y,Z)V)X + S(X,Y)W_8(\xi,Z)V - S(\xi,Y)W_8(X,Z)V + S(X,Z)W_8(Y,\xi)V - S(\xi,Z)W_8(Y,X)V + S(X,V)W_8(Y,Z)\xi - S(\xi,V)W_8(Y,Z)X = 0$

$\forall X,Y,Z,V \in \mathfrak{X}(M)$. (5.1)

Making inner product with ξ, the relation (5.1) becomes:
$S(X,W_8(Y,Z)V) - S(\xi,W_8(Y,Z)V)\eta(X) + S(X,Y)\eta(W_8(\xi,Z)V) - S(\xi,Y)\eta(W_8(X,Z)V) + S(X,Z)\eta(W_8(Y,\xi)V) - S(\xi,Z)\eta(W_8(Y,X)V) + s(X,V)\eta(W_8(Y,Z)\xi) - S(\xi,v)\eta(W_8(Y,Z)X)$

$\forall X,Y,Z,V \in \mathfrak{X}(M)$. (5.2)

Expanding the terms of (5.2) and using(3.5) in(3.6), we get:

$$(\mu+\lambda)\{\frac{1}{2n}S(X,Y)+\eta(X)\eta(Y)+g(X,Y)-\eta(X)\eta(Y)\}=0. \qquad (5.3)$$

Hence,

$$\frac{1}{2n}S(X,Y)=-g(X,Y) \qquad (5.4)$$

$$S(X,Y) = -2ng(X,Y). \qquad (5.5)$$

Thus, we have the following theorem

Theorem 5.1. If (ϕ,ξ,η,g) is a para- Kenmotsu structure on the $(2n + 1)$- dimensional manifold M, (g,ξ,η,λ,μ) is an η-Ricci soliton on M satisfying $(\xi,.)_S.W_8 = 0$, then $\mu = 0$ and $\lambda = 2n$.

Corollary 5.2. If (ϕ,ξ,η,g) is a para- Kenmotsu structure on the $(2n + 1)$- dimensional manifold M, (g,ξ,η,λ,μ) is an η-Ricci soliton on M satisfying $(\xi,.)_S.W_8 = 0$, then M is Einstein manifold.
Proof. From the above theorem, the Ricci tensor is collinear with metric. Hence by definition (3.1) we deduce the corollary.

Corollary 5.3. On a para-Kenmotsu manifold (M,ϕ,ξ,η,g)satisfying $(\xi,.)_S.W_8 = 0$, there is no Ricci solitons with potential vector field ξ.
Proof. The relation (5.4) implies that $L_\xi g = 0$ and (1.1) becomes:

$2S(X,y) + 2\lambda g(X,Y) = 0 \ \forall X,Y \in \mathfrak{X}(M)$ (5.6)

That is, $L_\xi g = 0$ and hence, g is invariant in direction ξ.

6. *η*− Ricci solitons on para- Kenmotsu manifolds having cyclic Ricci tensor
A Reimannian manifold (M,g) is said to have a cyclic Ricci tensor if:

$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$, (6.1)

Where:

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y,Z) - S(Y,\nabla_X Z). \tag{6.2}$$

Using (3.5), we get:

$$(\nabla_X S)(Y,Z) = -(\lambda + 1)[Xg(Y,Z) - g(\nabla_x Y,Z) - g(Y,\nabla_x Z)] \tag{6.3}$$

$$-(\mu - 1)[X\eta(Y)\eta(Z) - \eta(\nabla_X Y)\eta(Z) - \eta(Y)\eta(\nabla_X Z)].$$

As $\nabla g = 0$, (6.3) becomes:

$$(\nabla_X S)(Y,Z) = -(\mu - 1)[X\eta(Y)\eta(Z) - \eta(\nabla_X Y)\eta(Z) - \eta(Y)\eta(\nabla_X Z)]. \tag{6.4}$$

Hence,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = -(\mu - 1)[X\eta(Y)\eta(Z) - \eta(\nabla_X Y)\eta(Z) - \eta(Y)\eta(\nabla_X Z)+Y\eta(X)\eta(Z) - \eta(\nabla_Y Z)\eta(X) - \eta(Z)\eta(\nabla_Y X)+Z\eta(X)\eta(Y) - \eta(\nabla_Z X)\eta(Y) - \eta(X)\eta(\nabla_Z Y)] \tag{6.5}$$

After computations, the condition (6.1) becomes:

$$-2(\mu-1)[((g(X,Y)-\eta(X)\eta(Y))\eta(Z)+(g(X,Z)-\eta(X)\eta(Z))\eta(Y)+(g(Y,Z)-\eta(Y)\eta(Z))\eta(X)] = 0. \tag{6.6}$$

Putting $Z = \xi$, we get:

$$-2(\mu - 1)[g(X,Y) - \eta(X)\eta(Y)] = 0 \tag{6.7}$$

Hence, $\mu = 1$ and we state the following theorem.

Theorem 6.1 If (ϕ,ξ,η,g) is a para- Kenmotsu structure on $(2n+1)-$ dimensional manifold M, and manifold (M,g) has Cyclic Ricci tensor, then $\mu = 1$ and $\lambda = 2n - 1$.

Corollary 6.2 If (ϕ,ξ,η,g) is a para- Kenmotsu structure on $(2n+1)-$ dimensional manifold M, and manifold (M,g) has Cyclic Ricci tensor, then M is quasi- Einstein.
Proof. Using the same argument as in corollary (4.2), the result follows from the above theorem.

Corollary 6.3 On a para-Kenmotsu manifold (M,ϕ,ξ,η,g) having Ricci cyclic tensor, there is no Ricci solitons with potential vector field ξ.
Proof. The relation (3.1) is Ricci soliton if $\mu = 0$ and λ 6= 0 but $\mu = 0$ implies $n = -1$, that is $\lambda = -2$ and this is a contradiction as dimension of manifold is always positive and $\mu + \lambda = 2n$.

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