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Prime ideals and ideal symmetry in near- rings

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Abstract
The most interesting and active field of current research in mathematics is theory of near- ring due to its wide application in coding theory, group theory, geometry, cryptography and block- designing etc. Near rings are one of the generalized structure of rings. Modern ring theory has very active mathematical discipline and studying rings in their own right. In this paper we study about 3- prime ideals and radicals in a Near-ring. Also, we study about 3- prime ideal and ideal symmetry in a Near-ring and its inter relationship between them.

Keywords: Near- ring, 3- prime ideal, Ideal symmetry, semi- simple ideal symmetric graph

Introduction
The most interesting and active field of current research in mathematics is theory of near- ring due to its wide application in coding theory, group theory, geometry, cryptography and block- designing etc. Near rings are one of the generalized structure of rings, The study and research on near- rings is very systematic and continuous. Near- rings around in all directions of mathematics and continuous research is being conducted, which show that their structure has power and beauty all its own. Modern ring theory has very active mathematical discipline and studying rings in their own right. The key ideas important to near- ring were formed by L.E. Dickson in 1905. Actually he gave the concept of near- field. In 1930 Wieland studied near- rings which were not near- fields. In this paper we study about 3- prime ideals and radicals in a Near-ring. Also, we study about 3- prime ideal and ideal symmetry in a Near-ring and its inter relationship between them.

1- prime ideal
An ideal I of a near- ring N is called 3- prime ideal, if $a, b \in N$ and $aNb \subseteq I$ implies either $a \in I \; or \; b \in I$ and N is called $3 -$ prime near ring if $\{0\}$ is a $3 -$ prime ideal of N.

Example: Let $N = \{o, a, b, c\}$ be a near- ring with addition and multiplication are defined as:
$0 + 0 = 0,0 + a = a, 0 + b = b, 0 + c = c. a + 0 = a, a + a = 0, a + b = c, a + c = b, b + 0 = b, b + a = b, b + b = 0, b + c = a, c + 0 = c, c + a = c, c + b = a$ and $c + c = 0$.
$0.0 = 0, 0. a = 0, 0. b = 0, 0. c = 0a. 0 = a, a. a = a,$
$a. b = 0, a. c = a, b. 0 = b, b. a = b. b. b = b, b. c = b,$
$c. 0 = b, c. a = c, c. b = b, c. c = c.$
Then $\{o, a\}$ is a 3- prime ideal of N.

Proposition 1.1: Every equi-prime ideal is 3- prime ideal but every 3- prime near ring need not be an equi-prime near ring.

Note: Equi-prime ideal ⇒ 3- prime ideal-⇒ prime ideal.

Example: Let $(N, +, .)$ be any group with at least three elements. We define multiplication on N as:
$a. b = a, if \; b \neq 0 \; and$
$0, if \; b = 0$

Then $(N,+,.)$ is a zero- symmetric near- ring. Therefore for any non zero elements $x,y \in N \Rightarrow xny = x \neq 0$. Hence $xNy \neq 0$. This shows that N is 3- prime. Let 0 not element of $\{x,y\} \subseteq N$ such that $x \neq y$ then for each element n of N.
$xnx = x$, if $n \neq 0 = xny$ and
0, if $n = 0$
Thus $xny = xnx$ for all $n \in N$ but, $x \neq y$. Hence N is not an equi-prime near-ring.

Corollary 1.2: Let I and J be ideals of a near ring N such that $I \subseteq J$, then $G_{\{0\}}(N) \subseteq G_I(N) \subseteq G_J(N) \subseteq G_N(N)$.

Proof: Since $I \subseteq J$ so, $\{0\} \subseteq I \subseteq J \subseteq N$. Therefore, we have the set of edges in $G_{\{0\}}(N) \subseteq T$he set of edges in $G_I(N) \subseteq$ The set of edges in $G_J(N) \subseteq T$he set of edges in $G_N(N)$. Hence from these we have $G_{\{0\}}(N) \subseteq G_I(N) \subseteq G_J(N) \subseteq G_N(N)$.

Proposition 1.3: Let $G_{\{0\}}$ is the prime graph of a near ring N. If $|N| = n$, then $G_N(N)$ is the complete graph on n vertices K_n.

Proposition 1.4: Let N be an integral domain and simple near ring. Let I be an ideal of N. Then $G_I(N) = K_n$, or $G_I(N)$ is a rooted tree with root vertex 0.

Definition: Let G be a graph of with vertex set $V(G)$. A strong vertex cut of a graph G is a subset $S \subseteq V(G)$ such that $(G - S)$ is totally disconnected. The strong vertex connectivity of G is define as $(G) = min(n \geq 0$, where $K(G) =$ strong vertex connectivity of G there exists a strong vertex cut $S \leq V(G)$ such that $|S| = n$).

Theorem 1.5: Let I be an ideal of a near ring N. If I be $3-$ prime, then I is a strong vertex cut of $G_I(N)$. If I is a 3- semi-prime ideal of N and a strong vertex cut of $G_I(N)$, then I is $3-$ prime. Converse of this theorem does not hold if the condition that I is 3- semiprime is omitted.

Example: Consider the graph $G_I(V_4)$ where $I = \{0\}$. Here $\{0\}$ is a strong vertex cut of $G_I(Z_4)$, however $I = \{0\}$ is not a 3- prime ideal since $2N2 \subseteq \{0\} but$ $2 \notin I = \{0\}$.
Example: In the graph $G_I(N)$, where $N = \{0,a,b,c\}, I = \{0,a\}$ we can observe that I is a 3- prime ideal of N.
Example: In the graph $G_I(\mathbb{Z}_6), I = \{0\}$, we observe that $I = \{0\}$ is not a 3- prime ideal of $\mathbb{Z}_6$. Since $I = \{0\}$ is not a strong vertex cut.

Proposition 1.6: Let I be a minimal $3-$ prime ideal of a near-ring N, then $K(G_I(N)) = |I|$.

Proof: Let I be a minimal $3-$ prime ideal of a near-ring N that is $I \neq \{0\}$ and if there is an another 3-prime ideal J such that $\{0\} \subset J \subset I$ then either $J = \{0\}$ or $J = I$. Since I is a minimal 3-prime ideal so, I is a strong vertex cut of $G_I(N)$. If we remove I from $G_I(N)$, then it is a totally disconnected graph. Hence $K(G_I(N)) = |I|$.

Theorem 1.7: Let I be a $3-$ prime ideal of a near-ring N and x be a vertex in $G_I(N)$ if $deg(x) = deg(0)$, then $x \in I$.

Theorem 1.8: Let N be a zero- symmetric near ring N and x be a vertex in $G_I(N)$ if $x \in I$, then $deg(x) = deg(0)$.

Theorem 1.8: Let N be a zero- symmetric near ring N and I be a $3-$ prime ideal of N, then $x \in I$ if and only if $deg(x) = deg(0)$ in $G_I(N)$.

Theorem 1.9: Let |N| =n and I be an ideal of a near ring N then, if N is a zero- symmetric near ring and I is a minimal 3- prime ideal of N, then $|I| = K(G_I(N)) \leq \delta(G_I(N))$.
Where $\delta(G_I(N))$ = minimum degree of the vertex in $G_I(N)$.

2. Prime ideal and Ideal symmetry

Ideal symmetric graph: The graph $\left(G_I(N)\right)$is said to be ideal symmetric if for every pair of vertices x,y in $\left(G_I(N)\right)$ with an edge between them, either $deg(x) = deg(0)$ or, $deg(y) = deg(0)$.

Example: $\left(G_I(\mathbb{Z}_4)\right)$ is the ideal symmetry graph, where $I = \{0,2\} mod(4)$ and $\mathbb{Z}_4 = \{0,1,2,3\} mod(4)$.

Proposition 2.1: The ideal symmetry of $G_I(N)$ implies the symmetry determine by the automorphism group of $G_I(N)$.

Proposition 2.2: Let N be zero symmetric near- ring and I be a 3- semi-prime ideal of N and If I is a strong vertex cut $G_I(N)$ then, $G_I(N)$is an ideal symmetric graph.

Proof: Suppose N be zero- symmetric near- ring and I be a 3- semi prime ideal of N. Also if I be a strong vertex cut of $G_I(N)$. So, I is a 3- prime ideal of N. Hence $G_I(N)$ is an ideal symmetric graph.

Proposition 2.3: Let N be a commutative ring with unit element 1. Let I be completely semi-prime ideal of N. If $G_I(N)$is ideal symmetric, then I is completely -prime and I is a strong vertex cut of $G_I(N)$.

Corollary 2.4: If N is an equiprime near-ring, then $G_{\{0\}}(N)$is ideal symmetric.

Proof: Let N be an equiprime near- ring. This implies $\{0\}$ is an equiprime ideal of N and N is a zero- symmetric near ring. Since each equiprime ideal is a 3- prime ideal. Hence $G_{\{0\}}(N)$ is an ideal symmetric graph.

Proposition 2.5: Let N be a zero symmetric near ring and I is $3-$ prime ideal of N then $G_I(N)$ is an ideal symmetric graph.

Theorem 2.6: Let I be a 3- prime ideal of the near ring N, then, the group $G_I(N)$is bipartite if and only if $I = \{0\}$.

Proof: Let the graph $(G_I(N))$ be bipartite and $I \neq \{0\}$. Let $V_1 = V\{G_I(N) - I\}$and $V_2 = I$ be two vertex set of $(G_I(N))$, then there exists at least one element $0 \neq x \epsilon I$. Since I is a 3- prime ideal of N, therefore, I is a strong vertx cut of $(G_I(N))$ and $(G_I(N))$ is ideal symmetric graph. So, for any $x \in I$, deg(x) = deg(0). This implies that x is adjacent to each vertex of the graph $G_I(N)$. Therefore, it is also connected to 0. Hence $(G_I(N))$ is not a bipartite graph, which is a contradiction.
Conversely, let $I = \{0\}$ be a 3- prime ideal of N, therefore, I is a strong vertex cut of $(G_I(N))$ and 0 is the only vertex which is connected to each vertex of $(G_I(N))$. There is no other pair of vertices which are connected. So, there are two vertex sets of $V\{G_I(N) - I\}$ and I such that there is no edge between the vertices of $V\{G_I(N) - I\}$and also for I. Hence, the graph $(G_I(N))$ is bipartite.

Proposition 2.7: Suppose $G_I(N)$ is an ideal symmetric graph and I is 3- semiprime ideal of N. For every $x \in N$, deg(x) = deg(0) in $G_I(N)$ implies $x \in I$ then I is 3- prime and I is a strong vertex cut of $G_I(N)$.

Theorem 2.8: Let I be a 3-prime ideal of a near- ring N, then edge connectivity of $(G_I(N))$ is $|I|$.

Proof: Since I is a 3- prime ideal of N, then I is a strong vertex cut of $(G_I(N))$ and $(G_I(N))$ is ideal symmetric. Since I is 3- prime ideal this implies $xNy \nsubseteq I \Rightarrow x \in I$ or $y \in I$. Since, $(G_I(N))$ is ideal symmetric graph, this implies either deg(x) = deg(0) or deg(y) = deg(0). Also each element $x \epsilon I$ is connected to each element of N except x itself. Since I is 3- prime ideal of N, therefore, $xNy \nsubseteq I \Rightarrow x \notin I$ or $y \notin I$ This implies that there exists no edge between any pair (x, y) if $x, y \notin I$, So, for any $x \notin I$. $deg(x) = (|I|)$. Therefore, for any x in $(G_I(N))$ and $x \notin I$. Then $deg(x) = (|I|)$. Hence, edge connectivity of $(G_I(N)) = |I|$.

Theorem 2.9: Let I be a 3- prime ideal of a near ring N. If $I = \{0\}$, then $(G_I(N))$ is a planer graph. But converse need not be true.

Proof: Let I be a 3- prime ideal of a near ring N, then I is a strong vertex cut of $(G_I(N))$. Since 0 is the only vertex which is connected to each element of N except itself. Therefore, there is no pair of vertices $0 \neq x, 0 \neq y \in I$ such that there exists an edge between x and y. Hence, $(G_I(N))$ can be drawn on the plane in such a way that its edges intersect only at their end points that is it can be drawn in such a way that no edge cross to each other.

Theorem 2.10: Let I be a 3- prime ideal of a near ring N, then $(G_I(N))$ is not a cycle graph.

Proof: Given that I be a 3- prime ideal of a near ring N.
Case (1): If $I = \{0\}$, then $(G_I(N))$ is a star graph that is the zero vertex connected to each element of N except itself. Therefore, there is no cycle in $(G_I(N))$. Hence, $(G_I(N))$ is not a cycle graph.
Case (2): If $|N| < 3$, then there is no cycle in $(G_I(N))$. Hence, $(G_I(N))$ is not a cycle graph.
Case (3): If$|N| \geq 3$ and $I \neq \{0\}$ that is I has at least one $0 \neq x \in I$ and x is connected to each element of N except itself. Since$|N| \geq 3$, so, there are more than one 3- cycles. Hence $(G_I(N))$ is not a cycle graph.

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