International Journal of Statistics and Applied Mathematics 2019; 4(6): 48-52

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2019; 4(6): 48-52

www.mathsjournal.com
Received: 22-09-2019
Accepted: 24-10-2019

Collins Musera
Department of Statistics and Actuarial Science Maseno University, P.O Private Bag, Maseno, Kenya

Joseph Eyang`An Esekon
Department of Mathematics Kirinyaga University, P.O Box 75-10200, Kirinyaga

Onyango Fredrick
Department of Statistics and Actuarial Science Maseno University, P.O Private Bag, Maseno, Kenya

Corresponding Author:
Collins Musera
Department of Statistics and Actuarial Science Maseno University, P.O Private Bag, Maseno, Kenya

Accurate option values of a nonlinear black-scholes equation with price slippage

Collins Musera, Joseph Eyang`An Esekon and Onyango Fredrick

Abstract
Modern finance overlaps with many fields of mathematics, in particular, probability theory and stochastic processes. A nonlinear Black-Scholes Partial Differential Equation whose nonlinearity is as a result of transaction costs and a price slippage impact that lead to market illiquidity with feedback effects was studied. Most of the solutions obtained in option pricing especially using nonlinear equations are numerical which gives approximate option values. To get exact option values, analytic solutions for these equations have to be obtained. Analytic solutions to the nonlinear Black-Scholes Partial Differential Equation for pricing call and put options to expiry time are currently unknown. The main purpose of this paper was to obtain analytic solutions of European call and put options of a nonlinear Black-Scholes Partial Differential Equation with transaction costs and a price slippage impact. The methodology involved reduction of the equation into a second-order nonlinear Partial Differential Equation. By assumption of a traveling wave profile the equation was further reduced to Ordinary Differential Equations. Solutions to all the transformed equations gave rise to an analytic solution to the nonlinear Black-Scholes equation for a call option. Using the put-call parity relation the put option's value was obtained. The solutions obtained will be used to price put and call options in the presence of transaction costs and a price slippage impact. The solutions may also help in fitting the Black- Scholes option pricing model in the modern option pricing industry since it incorporates real world factors. In conclusion, it is clear that the time to expiry is the most critical for option-pricing models. The illiquid model in this paper admits to well-posed solutions of options on time to expiry because whilst remaining well-posed to expiry the option price behavior also remains sufficiently different from that of the corresponding liquid option. We have obtained efficient and accurate solutions for pricing European call and put options.

Keywords: Analytic solution, feedback effects, illiquid markets, transaction costs, price slippage, put-call parity

1. Introduction

In formulating classical arbitrage pricing theory, two primary assumptions are used: *frictionless* and *competitive markets*. In a frictionless market, there are no transaction costs and restrictions on trade. Restrictions on trade are imposed when we have extreme market conditions. In particular, short sales/purchases are not permitted when the market has shortage/surplus. A trader can buy or sell any quantity of a security without changing its price in a competitive market. The notion of liquidity risk is introduced on relaxing the assumptions above. The purpose of this paper was to obtain analytic solutions of the nonlinear Black-Scholes equation arising from transaction costs in the presence of price slippage impact by Backstein and Howison (2003) in [1].

To obtain the analytic solution for the options, the transformations $x = \ln\left(\frac{K}{S}\right)$ and $\tau = T - t$ were applied to the Black-Scholes Partial Differential Equation for modeling illiquid markets in the presence of transaction costs and a price slippage impact. Assuming a travelling wave solution to the resulting second-order nonlinear Partial Differential Equation (PDE) reduced it further to Ordinary Differential Equations (ODEs). All the transformed equations were solved to obtain an analytic solution to the nonlinear Black-Scholes Partial Differential Equation which is used for pricing a call option at $t \geq 0$. Using the put-call parity relation yielded the value of the put option. The justification of the transformation $x = \ln\left(\frac{K}{S}\right)$ is that it can be written as $S = Ke^{-x}$ and brings the notion of discounting.

The justification of the transformation $\tau = T - t$ is in order to allow the use of put-call parity relation as put-call relies on time to expiry.
This paper is organized as follows. Section 2 describes the nonlinear Black-Scholes PDE used for modelling illiquid markets with transaction costs and a price slippage impact. The smooth solutions to the nonlinear Black-Scholes equation for call and put options is presented in Section 3. Section 4 concludes the paper.

2. Backstein and Howison (2003) Equation

Considering the continuous-time feedback effects for illiquid markets, Backstein and Howison[1] used two assets in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$) whose value at time t is $B_t \equiv 1$, and a stock S. The stock was assumed to be risky and illiquid while the bond was assumed to be riskless and liquid. The equation (as in Theorem 3.1) of[1] is given by

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 V_{SS}^3 + rSV_S - rV = 0 \; (2.1)$$

where $\rho \geq 0$ is a measure of the liquidity of the market, σ is the volatility, $V_t = \frac{\partial V}{\partial t}$, $V_S = \frac{\partial V}{\partial S}$, $V_{SS} = \frac{\partial^2 V}{\partial S^2}$, t is time, r is the risk-free interest rate, V is the option price and α is the measure of price slippage impact felt by all participants of a market.
The solution for a Call option for equation (2.1) for time t is found in Theorem 3.2 of[2] after transformations. This solution, however, does not consider the value of the option up to time to expiry T.
Liquidity in equation (2.1) above has been defined through a combination of transaction costs and a price slippage impact. Due to ρ, bid-ask spreads dominate the price elasticity effect and when $\alpha = 1$, it corresponds to no slippage and the equation reduces to the PDE given by

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + rSV_S - rV = 0 \; (2.2)$$

whose solution for a Call option is found in Theorem 3.0.2 of [3] for $r > 0$ and Theorem 4.1 of[4] and in Theorem 3.2 of[5] for $r = 0$. The magnitude of the market impact is determined by ρS. Large ρ implies a big market impact of hedging.
If $\rho = 0$, the asset's price in equation (2.1) follows the standard Black-Scholes model in [6] with constant volatility σ. The terminal condition for a European call option is given by

$$h(S_T) = V(S,T) = max\{S_T - K, 0\} \text{ for } S \geq 0,$$

where T is the expiry time, $K > 0$ is the striking price and $h(S_T)$ is a terminal claim whose hedge cost $V(S_t, t)$ is the solution to (2.1).

The boundary conditions for the option are as follows:

$$V(0,t) = 0 \text{ for } 0 \leq t \leq T,\ V(S,t) \sim S - Ke^{-r(T-t)} \text{ as } S \sim \infty.$$

We take the last condition to mean that

$$\lim_{S\to\infty} \frac{V(S,t)}{S - Ke^{-r(T-t)}} = 1 \text{ uniformly for } 0 \leq t \leq T \text{ with the constraint } V(S,t) \geq 0.$$

The payoff profile for the put option is given by

$$h(S_T) = V^P(S,T) = \max\{K - S_T, 0\} \text{ for } S \geq 0$$

In order to price an option, one has to complete several steps:
1. Specify a suitable mathematical model describing sufficiently well the behavior of the stock market;
2. Calibrate the model to available market data;
3. Derive formula or equation for the price of the option of interest;
4. Compute the price of the option.

Backstein and Howison[1] only developed the model but did not solve it and can therefore not be used in computing option values as it is. This paper obtained analytic solutions for both call and put options considering time to expiry T so that the model can now be used for pricing this options.

3. Smooth Solutions to the Backstein and Howison (2003) Equation

3.1 Current Value of a Call Option

Lemma 3.1 *If $\upsilon(\zeta)$ is a twice continuously differentiable function, and x and τ are the spatial and time variables respectively, then there exists a traveling wave solution to the equation,*

$$U_\tau - \frac{1}{2}\sigma^2(U_{xx} - U_x)\big(1 + 2(U_{xx} - U_x)\big) - \frac{1}{2}(1-\alpha)^2\sigma^2(U_{xx} - U_x)^3 + rU_x = 0 \quad (3.1)$$

In $\mathbb{R} \times [0, \infty)$ *of the form*

$$U(x,\tau) = \upsilon(\zeta) \text{ where } \zeta = x - c\tau,\ \zeta \in \mathbb{R}; \tag{3.2}$$

for $0 \le \alpha < 1$,$1 < \alpha \le 2$, $r, \sigma \ge 0$, $\tau \ge 0$ *and* $x \in \mathbb{R}$ *such that* $U(x,\tau)$ *is a travelling wave of permanent form which translates to the right with constant speed* $c > 0$.

Proof

Partial differentiation of (3.2) gives

$U_\tau = -c\upsilon'(\zeta)$, $U_x = \upsilon'(\zeta)$, and $U_{xx} = \upsilon''(\zeta)$

where the prime denotes $\frac{d}{d\zeta}$. Substituting these expressions into (3.1) and rearranging, we conclude that $\upsilon(\zeta)$ must satisfy the nonlinear second order ODE

$$c\upsilon' + \tfrac{1}{2}\sigma^2(\upsilon'' - \upsilon')\big(1 + 2(\upsilon'' - \upsilon')\big) + \tfrac{1}{2}(1-\alpha)^2\sigma^2(\upsilon'' - \upsilon')^3 - r\upsilon' = 0 \tag{3.3}$$

in $\mathbb{R}$ and hence $U(x,\tau)$ solves (3.1) as required.
By setting $c = r$, the equation resulting from (3.3) can now be solved in a closed-form by first writing it as

$$(1-\alpha)^2(\upsilon'' - \upsilon')^2 + 2(\upsilon'' - \upsilon') + 1 = 0$$

in $\mathbb{R}$ where $(1-\alpha)^2 \neq 0$. The quadratic equation above is solved to get

$$\upsilon'' - \upsilon' = \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}, 0 \le \alpha < 1, 1 < \alpha \le 2\,.$$

Upon integration, we get the variable separable standard form see[7]

$$\upsilon' = e^{\zeta_0 + \zeta} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}, 0 \le \alpha < 1, 1 < \alpha \le 2\,, \zeta_0, \zeta \in \mathbb{R}$$

where ζ_0 is a constant of integration. This is the first order linear autonomous and separable ODE whose solution upon integration is given by

$$\upsilon(\zeta) = e^{\zeta_0 + \zeta} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\zeta + \psi$$

for $0 \le \alpha < 1, 1 < \alpha \le 2\,, \zeta_0, \zeta, \psi \in \mathbb{R}$, where ψ is another constant of integration.

Applying the initial condition

$\upsilon(0) = 0$
to the equation above and simplifying, gives

$\psi = -e^{\zeta_0}$.

Hence

$$\upsilon(\zeta) = e^{\zeta_0 + \zeta} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\zeta - e^{\zeta_0}$$

for $0 \le \alpha < 1, 1 < \alpha \le 2\,, \zeta_0, \zeta, \in \mathbb{R}$.

Since $U(x,\tau) = \upsilon(\zeta)$ *where* $x - r\tau = x - c\tau$, we obtain

$$U(x,\tau) = e^{x_0 + (x - r\tau)} - \frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}(x - r\tau) - e^{x_0} \quad (3.4)$$

for $0 \leq \alpha < 1, 1 < \alpha \leq 2$, $x_0, x \in \mathbb{R}$, $r > 0, \tau \geq 0$ since $\zeta_0 = x_0 - c \cdot 0 = x_0$ and $c = r$.

Theorem 3.2 *If $U(x,\tau)$ is any positive solution to the nonlinear equation*

$$U_\tau - \frac{1}{2}\sigma^2(U_{xx} - U_x)\big(1 + 2(U_{xx} - U_x)\big) - \frac{1}{2}(1-\alpha)^2\sigma^2(U_{xx} - U_x)^3 + rU_x = 0$$

in $\mathbb{R} \times [0,\infty)$ *then*

$$V^C = \frac{1}{\rho}\left[-\left(\frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\left\{\ln\left(\frac{K}{S}\right) - r\tau\right\} + \frac{K}{S_0}\right)S + \frac{K^2}{S_0}e^{-r(T-t)}\right] \quad (3.5)$$

in $\mathbb{R} \times [0,\infty)$ *solves the nonlinear Black-Scholes equation*

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 V_{SS}^3 + rSV_S - rV = 0 \quad (3.6)$$

Proof. To obtain the solution to (3.6) we apply the transformations

$$x = \ln\left(\frac{K}{S}\right)$$

$$U(x,\tau) = \rho\frac{V(S,t)}{Ke^{-x}}$$

And

$$\tau = T - t$$

To get

$$V_t = -\frac{S}{\rho}U_\tau,$$

$$V_S = \frac{1}{\rho}(U - U_x),$$

$$V_{SS} = \frac{1}{\rho S}(U_{xx} - U_x)$$

Substituting these expressions into (3.6) gives the equation in theorem (3.2). Hence applying the transformations above into (3.4) gives (3.5).

3.2 Put-Call Parity Relation

The Put-Call parity relation is given by

$$V^C + Ke^{-r(T-t)} = V^P + S \quad (3.7)$$

Where V^C is the current call option value, V^P is the current put option value, $e^{-r(T-t)}$is the discount factor, K is the strike price and S is the stock price. Using this relation, the value of a put option is given by

$$V^P = V^C + Ke^{-r(T-t)} - S \quad (3.8)$$

3.3 Current Value of a Put Option

Corollary 1 *If $U(x,\tau)$ is any positive solution to the nonlinear equation*

$$U_\tau - \frac{1}{2}\sigma^2(U_{xx} - U_x)\big(1 + 2(U_{xx} - U_x)\big) - \frac{1}{2}(1-\alpha)^2\sigma^2(U_{xx} - U_x)^3 + rU_x = 0$$

In $\mathbb{R} \times [0,\infty)$ *then*

$$V^P = \frac{1}{\rho}\left[-\left(\frac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\left\{\ln\left(\frac{K}{S}\right) - r\tau\right\} + \frac{K}{S_0}\right)S + \frac{K^2}{S_0}e^{-r(T-t)}\right] + Ke^{-r(T-t)} - S \quad (3.9)$$

In $\mathbb{R} \times [0, \infty)$ solves the nonlinear Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS}(1 + 2\rho S V_{SS}) + \frac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 V_{SS}^3 + rSV_S - rV = 0 \ (3.10)$$

For $r, K, S, S_0, \rho > 0, \tau \geq 0, 0 \leq \alpha < 1$ and $1 < \alpha \leq 2$, S_0 is the initial stock price.

Proof.
To obtain the solution to equation (3.10) the procedure is same as in theorem (3.2) using equation (3.8) and the transformations $x = \ln\left(\frac{K}{S}\right)$ and $\tau = T - t$.

4. Conclusion

The interplay between finance and mathematics is very challenging and often stimulates the use of innovative mathematics and computational mathematics techniques. In this paper, we have built on the work of Backstein and Howison (2003). We have obtained analytic solutions of European call and put options of nonlinear Black- Scholes Equation in the presence of transaction costs and a price slippage impact that lead to market illiquidity with feedback effects. Assuming the solution of a forward wave, classical solutions for the nonlinear Black-Scholes equation were found. The solutions obtained can be used for pricing European call and put options at time $t \geq 0$. After obtaining the value of a call option, put-call parity relation was used to obtain the put option's value at $t \geq 0$.

The solutions obtained in this paper supports the comments in [3, 4, 6]. It is clear that the time to expiry is the most critical for option-pricing models. The Bakstein and Howison model [1] admits to well-posed solutions of options on time to expiry because firstly, whilst remaining well-posed close to expiry the option price behavior also remains sufficiently different from that of the corresponding Black-Scholes (liquid) option. Secondly, in the limit of no price slippage, this model reduces to the model of Cetin *et al.*, [8] which has become a popular model for liquidation costs in recent years. We have obtained efficient and accurate solutions for pricing European call and put options in the presence of transaction costs and price slippage.

5. References

1. Bakstein D, Howison S. A non-arbitrage liquidity model with observable parameters for derivatives. Working paper, 2003.
2. Esekon JE. Analytic solution of a nonlinear Black-Scholes equation with price slippage. IJPAM, 2014; 96(2):229-234.
3. Esekon JE. Analytic solution of a nonlinear Black-Scholes equation. IJPAM, 2013; 82(3):547-555,
4. Esekon JE, Onyango S, Omollo N. Analytic solution of a nonlinear Black-Scholes partial differential equation. IJPAM, 2010; 61(2):219-230,
5. Esekon JE. A particular solution of a nonlinear Black-Scholes partial differential equation. IJPAM, 2012; 81(5):715-721,
6. Black F, Scholes M. The pricing of options and corporate liabilities. J polit. Econ. 1973; 81:637-654,
7. Logan JD. An Introduction to Nonlinear Partial Differential Equations, John Wiley and Sons, USA, 2008,
8. Cetin U, Jarrow R, Potter P. Liquidity risk and Arbitrage pricing theory. Finance Stoch. 2004; 8:311-341.
9. P. Athanasios. Probability, Random variables and stochastic processes. Mc-Graw
10. Hill, Third edition, 1991.