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Numerical application of third derivative hybrid block methods on third order initial value problem of ordinary differential equations

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Abstract
The numerical application of third derivative on third order initial value problem of ordinary differential equations is consider in this paper. The method is derived by collocating and interpolating the approximate solution in power series, while Taylor series is used to generate the independent solution at selected grid and off grid points. The basic analysis of the method were established and it was found to be consistent, zero-stable and convergent. The developed method is then applied to solve some third order initial value problems of ODEs, and the result computed shows that the derived method is more accurate than some existing methods considered in this paper. We further plotted the solution graph of each problems and it is obvious that the numerical solution convergence toward the exact solution.

Keywords: Numerical application, third derivative, power series, taylor series, IVP of ODEs

Introduction

Many problems in Physics, Chemistry, and Engineering science are demonstrated mathematically by third-order boundary value problems or initial value problems. These boundary value problems can be found in different areas of applied mathematics and physics as, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves, or gravity driven flows. Third-order boundary value problems were discussed in many papers in recent years Ejaz and Mustafa [1], Areo and Omojola [2], Adeyeye and Omar [3], Sunday [4], Fsasis [5], Omar and Adeyeye [6], Awoyemi, Kayode and Adeghe [7], etc.

Ordinary differential equations are widely applied to model real life situations particularly involving engineering problems. The third order initial value problems of ordinary differential equations of the form

$$y''' = f(x, y, y', y''), \; y(x_0) = y_0, \; y'(x_0) = y'_0, \; y''(x_0) = y''_0 \tag{1.1}$$

is considered in this paper. The conventional approach for solving higher order initial value problems is by reducing them to their equivalent systems of first order ordinary differential equations and then solving by using the existing methods. This is widely discussed in Goult, *et al.* [8], Lambert [9], Lambert and Watson [10], Fatunla [11] and Awoyemi [12]. However, this approach has some setbacks such as burden of computing and difficulties in designing computer programme which may have affect the accuracy of the method in terms of error.

Direct method was implemented in different ways such as predictor-corrector method, block linear multistep method and hybrid block method. Hybrid block method was introduced to combine the advantages of block method and overcoming the zero stability barrier in linear multistep method, Skwame, *et al.* [13, 14]. This barrier implies that the highest order of zero stability for linear multistep method of step length k is k+2 when k is even and k+1 when k is odd, Lambert [9].

Omar, Abdullahi and Kuboye [15] proposed a Predictor-Corrector block method of order seven for solving third order ordinary differential equations.

Application of single step with three generalized hybrid points block method for solving third order ordinary differential equations has been studied by Omar and Abdelrahim [16]. And Ogunware, *et al*., [17] developed a numerical treatment of general third order ordinary differential equations using Taylor series as predictor, among others are Odunkunle, *et al*., [18], Ejaz and Mustafa [1], Kubuye and Omar [19]. In this work, we develop one step hybrid block method with equal interval for solving (1.1) directly. This paper describes the development, analysis and implementation of one-step hybrid block method for solving third order initial value problems of ordinary differential equations directly using Taylor Series approximation method.

2. Formation of the Bloch methods

We consider the general power series as basic function for approximation

$$Y(x)=\sum_{j=0}^{r+s-1}a_j\varphi(x)_j \tag{2.1}$$

Where $\varphi(x)=x^j$ and $x\in[a,b]$, $a_j s$ are coefficient to be determined and is a polynomial of degree $r+s-1$. We construct a K-step collocation method by imposing the following conditions on (2.1)

$$Y(x_{n+j})=y_{n+j},\quad j=0,1,2,\cdots,r-1 \tag{2.2}$$

$$Y(x_{n+j})=y_{n+j},\quad j=0,1,2,\cdots,s-1 \tag{2.3}$$

Differentiating (2.1) three times, yield

$$Y'''(x)=\sum_{j=0}^{9}j(j-1)(j-2)a_jx^{j-3} \tag{2.4}$$

Substituting (2.4) into equation (2.1) to give

$$y(q,y,y'\,y'')=\sum_{j=0}^{9}j(j-1)(j-2)a_jx^{j-3} \tag{2.5}$$

Now a one-step method is developed in this paper with equal equidistance of

$t_{n+\frac{1}{5}},t_{n+\frac{2}{5}},t_{n+\frac{3}{5}},t_{n+\frac{4}{5}}$ and $h=t_{n+i}-t_n,\ i=0,1$.

Interpolating (2.1) at a points $t_{n+s},\ s=0,\frac{4}{5},1$ and collocate (2.5) at a points $t_{n+r},\ r=0\left(\frac{1}{5}\right)1$, which yield a system of nonlinear equation of the form,

$$TA=U \tag{2.6}$$

Where

$$A=\begin{bmatrix}a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8\end{bmatrix}^T$$

$$U=\begin{bmatrix}y_n & y_{n+\frac{3}{5}} & y_{n+\frac{4}{5}} & y_{n+1} & f_{n+\frac{1}{5}} & f_{n+\frac{2}{5}} & f_{n+\frac{3}{5}} & f_{n+\frac{4}{5}} & f_{n+1}\end{bmatrix}^T$$

And

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+\frac{4}{5}} & x^2_{n+\frac{4}{5}} & x^3_{n+\frac{4}{5}} & x^4_{n+\frac{4}{5}} & x^5_{n+\frac{4}{5}} & x^6_{n+\frac{4}{5}} & x^7_{n+\frac{4}{5}} & x^8_{n+\frac{4}{5}} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\ 0 & 0 & 0 & 6 & 24x_n & 60x^2_{n+\frac{1}{5}} & 120x^3_{n+\frac{1}{5}} & 210x^4_{n+\frac{1}{5}} & 336x^5_{n+\frac{1}{5}} \\ 0 & 0 & 0 & 6 & 24x_n & 60x^2_{n+\frac{2}{5}} & 120x^3_{n+\frac{2}{5}} & 210x^4_{n+\frac{2}{5}} & 336x^5_{n+\frac{2}{5}} \\ 0 & 0 & 0 & 6 & 24x_n & 60x^2_{n+\frac{3}{5}} & 120x^3_{n+\frac{3}{5}} & 210x^4_{n+\frac{3}{5}} & 336x^5_{n+\frac{3}{5}} \\ 0 & 0 & 0 & 6 & 24x_n & 60x^2_{n+\frac{4}{5}} & 120x^3_{n+\frac{4}{5}} & 210x^4_{n+\frac{4}{5}} & 336x^5_{n+\frac{4}{5}} \\ 0 & 0 & 0 & 6 & 24x_n & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 \end{bmatrix}$$

Solving (2.6) for $a_j,\ j=0(1)8$ which will be determined and are constants, on putting the result back into (2.1) and simplifying gives a continuous linear multistep method of the form

$$Y(x)=\alpha_0(x)y_n+\alpha_{\frac{4}{5}}(x)y_{n+\frac{4}{5}}+\alpha_1(x)y_{n+1}+h^3\left[\sum_{j=0}^{1}\beta_j(x)f_{n+j}\right],\quad q=\frac{1}{5}\left(\frac{1}{5}\right)\frac{4}{5} \tag{2.7}$$

Where $\alpha_s(t),\beta_j(t)$ and $\beta_q(t)$ are the coefficient to be determined and are expressed as function of χ with continuous schemes take the following form

$$\left.\begin{aligned}
&\alpha_0=4+5x^2-9x,\quad \alpha_{\frac{4}{5}}=25x-25x^2,\quad \alpha_1=-4t+5x^2\\
&\beta_0=\frac{48}{16800}xh^3-\frac{6341}{201600}x^2h^3+\frac{1}{6}x^3h^3-\frac{137}{288}x^4h^3+\frac{25}{32}x^5h^3-\frac{425}{576}x^6h^3+\frac{125}{336}x^7h^3-\frac{625}{8064}x^8h^3\\
&\beta_{\frac{1}{5}}=\frac{241}{7200}xh^3-\frac{9041}{67200}x^2h^3+\frac{25}{24}x^4h^3-\frac{385}{144}x^5h^3+\frac{1775}{576}x^6h^3-\frac{125}{72}x^7h^3+\frac{3125}{8064}x^8h^3\\
&\beta_{\frac{2}{5}}=\frac{1021}{25200}xh^3-\frac{1403}{33600}x^2h^3-\frac{25}{24}x^4h^3-\frac{535}{144}x^5h^3+\frac{1475}{288}x^6h^3+\frac{1625}{504}x^7h^3-\frac{3125}{4032}x^8h^3\\
&\beta_{\frac{3}{5}}=\frac{353}{8400}xh^3-\frac{8111}{100800}x^2h^3+\frac{25}{36}x^4h^3-\frac{65}{24}x^5h^3+\frac{1225}{288}x^6h^3-\frac{125}{42}x^7h^3+\frac{3125}{4032}x^8h^3\\
&\beta_{\frac{4}{5}}=\frac{29}{2016}xh^3-\frac{9}{896}x^2h^3-\frac{25}{96}x^4h^3-\frac{305}{288}x^5h^3-\frac{1025}{576}x^6h^3+\frac{1375}{1008}x^7h^3-\frac{3125}{8064}x^8h^3\\
&\beta_1=\frac{19}{50400}xh^3-\frac{39}{22400}x^2h^3+\frac{1}{24}x^4h^3-\frac{25}{144}x^5h^3+\frac{175}{576}x^6h^3-\frac{125}{504}x^7h^3+\frac{625}{8064}x^8h^3
\end{aligned}\right\} \tag{2.8}$$

on differentiating (2.7)

$$\left.\begin{aligned}
&\sigma_0=-\frac{6341}{100800}h^3+xh^3+\frac{137}{24}x^2h^3+\frac{125}{8}x^3h^3-\frac{2125}{96}x^4h^3+\frac{125}{8}x^5h^3-\frac{625}{144}x^6h^3\\
&\sigma_{\frac{1}{5}}=-\frac{9041}{33600}h^3+\frac{25}{2}x^2h^3-\frac{1925}{36}x^3h^3+\frac{8875}{8}x^4h^3-\frac{875}{12}x^5h^3+\frac{3125}{144}x^6h^3\\
&\sigma_{\frac{2}{5}}=-\frac{1403}{16800}h^3-\frac{25}{2}x^2h^3+\frac{2675}{36}x^3h^3-\frac{7375}{48}x^4h^3+\frac{1625}{12}x^5h^3-\frac{3125}{72}x^6h^3\\
&\sigma_{\frac{3}{5}}=-\frac{8111}{50400}h^3+\frac{25}{3}x^2h^3-\frac{325}{6}x^3h^3+\frac{6125}{48}x^4h^3-125x^5h^3+\frac{3125}{72}x^6h^3\\
&\sigma_{\frac{4}{5}}=-\frac{9}{448}h^3-\frac{25}{8}x^2h^3+\frac{1525}{72}x^3h^3-\frac{5125}{96}x^4h^3+\frac{1375}{24}x^5h^3-\frac{3125}{144}x^6h^3\\
&\sigma_1=-\frac{39}{11200}h^3+\frac{1}{2}x^2h^3-\frac{125}{36}x^3h^3+\frac{875}{96}x^4h^3-\frac{125}{12}x^5h^3+\frac{625}{144}x^6h^3
\end{aligned}\right\} \tag{2.9}$$

On collecting and simplifying the continuous form of schemes

$$\left.\begin{aligned}
&y_{n+\frac{1}{5}}=\frac{3}{5}y_n+y_{+\frac{4}{5}}-\frac{3}{5}y_{n+1}+\frac{1}{30000}h^3f_n+\frac{23}{10000}h^3f_{n+\frac{1}{5}}+\frac{17}{3000}h^3f_{n+\frac{2}{5}}+\frac{17}{3000}h^3f_{n+\frac{3}{5}}\\
&+\frac{23}{10000}h^3f_{n+\frac{4}{5}}+\frac{1}{10000}h^3f_{n+1}\\
&y_{n+\frac{2}{5}}=\frac{3}{10}y_n+\frac{3}{2}y_{+\frac{4}{5}}-\frac{4}{5}y_{n+1}+\frac{1}{60000}h^3f_n+\frac{71}{60000}h^3f_{n+\frac{1}{5}}+\frac{47}{10000}h^3f_{n+\frac{2}{5}}+\\
&\frac{211}{30000}h^3f_{n+\frac{3}{5}}+\frac{181}{60000}h^3f_{n+\frac{4}{5}}+\frac{1}{20000}h^3f_{n+1}\\
&y_{n+\frac{3}{5}}=\frac{1}{10}y_n+\frac{3}{2}y_{+\frac{4}{5}}-\frac{3}{5}y_{n+1}+\frac{13}{30000}h^3f_{n+\frac{1}{5}}+\frac{11}{7500}h^3f_{n+\frac{2}{5}}+\frac{19}{5000}h^3f_{n+\frac{3}{5}}\\
&+\frac{17}{750}h^3f_{n+\frac{4}{5}}+\frac{1}{30000}h^3f_{n+1}
\end{aligned}\right\} \quad (2.10)$$

Evaluating y' at $x=x_{n+j},\quad j=0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1$, we obtain

$$\left.\begin{aligned}
&hy'_n=\frac{9}{4}y_n+\frac{25}{4}y_{+\frac{4}{5}}-4y_{n+1}+\frac{43}{16800}h^3f_n+\frac{241}{720}h^3f_{n+\frac{1}{5}}+\frac{1021}{25200}h^3f_{n+\frac{2}{5}}+\frac{353}{8400}h^3f_{n+\frac{3}{5}}\\
&+\frac{29}{2016}h^3f_{n+\frac{4}{5}}+\frac{19}{50400}h^3f_{n+1}\\
&hy'_{n+\frac{1}{5}}=-\frac{7}{4}y_n+\frac{15}{4}y_{+\frac{4}{5}}-2y_{n+1}-\frac{127}{504000}h^3f_n-\frac{541}{168000}h^3f_{n+\frac{1}{5}}+\frac{2957}{252000}h^3f_{n+\frac{2}{5}}+\\
&\frac{623}{36000}h^3f_{n+\frac{3}{5}}+\frac{429}{56000}h^3f_{n+\frac{4}{5}}+\frac{53}{50400}h^3f_{n+1}\\
&hy'_{n+\frac{2}{5}}=-\frac{5}{4}y_n+\frac{5}{4}y_{+\frac{4}{5}}-\frac{1}{1575}h^3f_n-\frac{8}{1575}h^3f_{n+\frac{1}{5}}-\frac{43}{2625}h^3f_{n+\frac{2}{5}}-\frac{8}{1570}h^3f_{n+\frac{3}{5}}-\\
&\frac{1}{15750}h^3f_{n+\frac{4}{5}}\\
&hy'_{n+\frac{3}{5}}=-\frac{4}{4}y_n-\frac{5}{4}y_{+\frac{4}{5}}+2y_{n+1}-\frac{1}{24000}h^3f_n-\frac{1481}{504000}h^3f_{n+\frac{1}{5}}-\frac{613}{50400}h^3f_{n+\frac{2}{5}}-\\
&\frac{657}{28000}h^3f_{n+\frac{3}{5}}+\frac{4009}{504000}h^3f_{n+\frac{4}{5}}-\frac{53}{50400}h^3f_{n+1}\\
&hy'_{n+\frac{4}{5}}=-\frac{1}{4}y_n-\frac{15}{4}y_{+\frac{4}{5}}+4y_{n+1}-\frac{1}{50400}h^3f_n-\frac{27}{28000}h^3f_{n+\frac{1}{5}}-\frac{71}{18000}h^3f_{n+\frac{2}{5}}-\\
&\frac{1199}{126000}h^3f_{n+\frac{3}{5}}-\frac{199}{16800}h^3f_{n+\frac{4}{5}}-\frac{19}{50400}h^3f_{n+1}\\
&hy'_{n+1}=\frac{1}{4}y_n-\frac{25}{4}y_{+\frac{4}{5}}+6y_{n+1}+\frac{17}{100800}h^3f_n+\frac{1}{100800}h^3f_{n+\frac{1}{5}}+\frac{37}{5600}h^3f_{n+\frac{2}{5}}\\
&+\frac{257}{50400}h^3f_{n+\frac{3}{5}}+\frac{11}{10000}h^3f_{n+\frac{4}{5}}+\frac{79}{33600}h^3f_{n+1}
\end{aligned}\right\} \quad (2.11)$$

Evaluating the second derivative of y' at $x = x_{n+j}, \quad j = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$, we obtain

$$
\left.
\begin{aligned}
&h^2 y''_n = \frac{5}{2}y_n - \frac{25}{2}y_{+\frac{4}{5}} + 10y_{n+1} - \frac{6341}{100800}h^3 f_n - \frac{9041}{33600}h^3 f_{n+\frac{1}{5}} - \frac{1403}{16800}h^3 f_{n+\frac{2}{5}} - \frac{8111}{50400}h^3 f_{n+\frac{3}{5}} \\
&- \frac{9}{448}h^3 f_{n+\frac{4}{5}} - \frac{39}{11200}h^3 f_{n+1} \\
&h^2 y''_{n+\frac{1}{5}} = \frac{5}{2}y_n - \frac{25}{2}y_{+\frac{4}{5}} + 10y_{n+1} + \frac{103}{33600}h^3 f_n - \frac{1429}{20160}h^3 f_{n+\frac{1}{5}} - \frac{653}{3360}h^3 f_{n+\frac{2}{5}} - \frac{1579}{16800}h^3 f_{n+\frac{3}{5}} \\
&- \frac{4447}{100800}h^3 f_{n+\frac{4}{5}} + \frac{3}{11200}h^3 f_{n+1} \\
&h^2 y''_{n+\frac{2}{5}} = \frac{5}{2}y_n - \frac{25}{2}y_{+\frac{4}{5}} + 10y_{n+1} - \frac{23}{33600}h^3 f_n + \frac{197}{11200}h^3 f_{n+\frac{1}{5}} - \frac{2641}{50400}h^3 f_{n+\frac{2}{5}} - \frac{727}{5600}h^3 f_{n+\frac{3}{5}} \\
&- \frac{1123}{33600}h^3 f_{n+\frac{4}{5}} - \frac{127}{100800}h^3 f_{n+1} \\
&h^2 y''_{n+\frac{3}{5}} = \frac{5}{2}y_n - \frac{25}{2}y_{+\frac{4}{5}} + 10y_{n+1} + \frac{17}{20160}h^3 f_n + \frac{157}{33600}h^3 f_{n+\frac{1}{5}} + \frac{991}{16800}h^3 f_{n+\frac{2}{5}} - \frac{929}{50400}h^3 f_{n+\frac{3}{5}} \\
&- \frac{519}{11200}h^3 f_{n+\frac{4}{5}} + \frac{3}{11200}h^3 f_{n+1} \\
&h^2 y''_{n+\frac{4}{5}} = \frac{5}{2}y_n - \frac{25}{2}y_{+\frac{4}{5}} + 10y_{n+1} - \frac{23}{33600}h^3 f_n + \frac{1549}{100800}h^3 f_{n+\frac{1}{5}} + \frac{389}{16800}h^3 f_{n+\frac{2}{5}} + \frac{83}{672}h^3 f_{n+\frac{3}{5}} \\
&+ \frac{4247}{100800}h^3 f_{n+\frac{4}{5}} - \frac{39}{11200}h^3 f_{n+1} \\
&h^2 y''_{n+1} = \frac{5}{2}y_n - \frac{25}{2}y_{+\frac{4}{5}} + 10y_{n+1} + \frac{103}{33600}h^3 f_n - \frac{97}{11200}h^3 f_{n+\frac{1}{5}} + \frac{4541}{50400}h^3 f_{n+\frac{2}{5}} + \frac{71}{5600}h^3 f_{n+\frac{3}{5}} \\
&+ \frac{323}{1344}h^3 f_{n+\frac{4}{5}} + \frac{6299}{100800}h^3 f_{n+1}
\end{aligned}
\right\} \tag{2.12}
$$

On simplifying (2.10), (2.11) and (2.12), gives the new formula

$$
\begin{pmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \end{pmatrix}
= y_n + h y'_n \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix}
+ h^2 y''_n \begin{pmatrix} \frac{1}{50} \\ \frac{2}{25} \\ \frac{9}{50} \\ \frac{8}{5} \\ \frac{1}{2} \end{pmatrix}
+ h^3 f_n \begin{pmatrix} \frac{3929}{5040000} \\ \frac{317}{78750} \\ \frac{783}{80000} \\ \frac{712}{39375} \\ \frac{233}{8064} \end{pmatrix}
+ h^3 \begin{pmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix}
\begin{pmatrix}
\frac{199}{201600} & \frac{367}{39375} & \frac{16119}{560000} & \frac{2336}{39375} & \frac{815}{8064} \\
-\frac{1931}{2520000} & -\frac{38}{7875} & -\frac{2187}{280000} & -\frac{32}{5625} & \frac{5}{4032} \\
\frac{173}{360000} & \frac{122}{39375} & \frac{423}{56000} & \frac{704}{39375} & \frac{155}{4032} \\
-\frac{883}{5040000} & -\frac{89}{78750} & -\frac{1539}{560000} & -\frac{8}{1575} & -\frac{5}{1152} \\
\frac{139}{5040000} & \frac{1}{5625} & \frac{243}{560000} & \frac{32}{39375} & \frac{11}{8064}
\end{pmatrix} \tag{2.13}
$$

$$\begin{pmatrix} y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1} \end{pmatrix} = y'_n + h y''_n \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} + h^2 f_n \begin{pmatrix} \frac{1231}{126000} \\ \frac{173}{7875} \\ \frac{123}{3500} \\ \frac{376}{7875} \\ \frac{61}{1008} \end{pmatrix} + h^2 \begin{pmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \begin{pmatrix} \frac{863}{50400} & \frac{544}{7875} & \frac{3501}{2800} & \frac{1424}{7875} & \frac{475}{2016} \\ -\frac{761}{63000} & -\frac{37}{1575} & -\frac{9}{3500} & \frac{176}{7875} & \frac{25}{504} \\ \frac{941}{126000} & \frac{136}{7875} & \frac{87}{2800} & \frac{608}{7875} & \frac{125}{1008} \\ -\frac{341}{126000} & -\frac{101}{15750} & -\frac{9}{875} & -\frac{16}{1575} & \frac{25}{1008} \\ \frac{107}{252000} & \frac{8}{7875} & \frac{9}{5600} & \frac{16}{7875} & \frac{11}{2016} \end{pmatrix} \tag{2.14}$$

$$\begin{pmatrix} y''_{n+\frac{1}{5}} \\ y''_{n+\frac{2}{5}} \\ y''_{n+\frac{3}{5}} \\ y''_{n+\frac{4}{5}} \\ y''_{n+1} \end{pmatrix} = y''_n + h f_n \begin{pmatrix} \frac{19}{288} \\ \frac{14}{225} \\ \frac{51}{800} \\ \frac{14}{225} \\ \frac{19}{288} \end{pmatrix} + h \begin{pmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix} \begin{pmatrix} \frac{1427}{7200} & \frac{43}{150} & \frac{219}{800} & \frac{64}{225} & \frac{25}{96} \\ -\frac{133}{1200} & \frac{7}{225} & \frac{57}{400} & \frac{8}{75} & \frac{25}{144} \\ \frac{241}{3600} & \frac{7}{225} & \frac{57}{400} & \frac{64}{225} & \frac{25}{144} \\ -\frac{173}{7200} & -\frac{1}{75} & -\frac{21}{800} & \frac{14}{225} & \frac{25}{96} \\ \frac{3}{800} & \frac{1}{450} & \frac{3}{800} & 0 & \frac{19}{288} \end{pmatrix} \tag{2.15}$$

3. Analysis of the basic properties of the method

3.1 Order and Error constant of the Method

Let the linear operator $L\{y(x):h\}$ defined by

$$L\{y(x):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{k} \frac{jh^{(i)}}{i!} y_n^{(i)} - h^{(3-1)}\left[d_i f(y_n) + b_i F(Y_m)\right], \tag{2.16}$$

Expanding Y_m *and* $F(Y_m)$ in Taylor series and comparing the coefficients of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots \tag{2.17}$$

Definition 3.1: The linear operator L and the associate block method are said to be of order p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = C_{p+2} = 0, \quad C_{p+3} \neq 0.$ C_{p+3} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+3} h^{p+3} y^{p+3}(x) + 0h^{p+4}$

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\left(\frac{1}{5}\right)^j}{j!}-y_n-\frac{1}{5}hy'_n-\frac{1}{50}h^2y''_n-\frac{3909}{5040000}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{199}{201600}\left(\frac{1}{5}\right)-\frac{1931}{2520000}\left(\frac{2}{5}\right)+\frac{173}{360000}\left(\frac{3}{5}\right)\\-\frac{883}{5040000}\left(\frac{4}{5}\right)+\frac{139}{5040000}(1)\end{bmatrix} \\ \sum_{j=0}^{\infty}\frac{\left(\frac{2}{5}\right)^j}{j!}-y_n-\frac{2}{5}hy'_n-\frac{2}{25}h^2y''_n-\frac{317}{78750}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{367}{39375}\left(\frac{1}{5}\right)-\frac{38}{7875}\left(\frac{2}{5}\right)+\frac{122}{39375}\left(\frac{3}{5}\right)\\-\frac{89}{78750}\left(\frac{4}{5}\right)+\frac{1}{5625}(1)\end{bmatrix} \\ \sum_{j=0}^{\infty}\frac{\left(\frac{3}{5}\right)^j}{j!}-y_n-\frac{3}{5}hy'_n-\frac{9}{50}h^2y''_n-\frac{783}{80000}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{16119}{560000}\left(\frac{1}{5}\right)-\frac{2187}{280000}\left(\frac{2}{5}\right)+\frac{142373}{56000}\left(\frac{3}{5}\right)\\-\frac{1539}{560000}\left(\frac{4}{5}\right)+\frac{243}{560000}(1)\end{bmatrix} \\ \sum_{j=0}^{\infty}\frac{\left(\frac{4}{5}\right)^j}{j!}-y_n-\frac{4}{5}hy'_n-\frac{8}{5}h^2y''_n-\frac{712}{39375}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{2336}{39375}\left(\frac{1}{5}\right)-\frac{32}{5625}\left(\frac{2}{5}\right)+\frac{704}{39375}\left(\frac{3}{5}\right)\\-\frac{8}{1575}\left(\frac{4}{5}\right)+\frac{32}{39375}(1)\end{bmatrix} \\ \sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-hy'_n-\frac{1}{2}h^2y''_n-\frac{5}{4032}h^3y'''_n-\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\begin{bmatrix}\frac{815}{8064}\left(\frac{1}{5}\right)+\frac{5}{4032}\left(\frac{2}{5}\right)+\frac{155}{4032}\left(\frac{3}{5}\right)\\-\frac{5}{1052}\left(\frac{4}{5}\right)+\frac{11}{8064}(1)\end{bmatrix} \end{bmatrix}$$

Comparing the coefficient of h, according to Sunday [4], the order p of our method (2.13) and the error constant are given respectively by $p=[4 \quad 4 \quad 4 \quad 4 \quad 4]^T$ and $C_{p+3}=[-1.8265\times10^{-7} \quad -1.2529\times10^{-7} \quad -1.6571\times10^{-7} \quad -1.0836\times10^{-7} \quad -12.9101\times10^{-7}]$

3.2 Consistency of the Method

A numerical method is said to be consistent if the following conditions are satisfied

The order of the method must be greater than or equal to zero to one i.e. $p\geq1$.

$$\sum_{j=0}^{k}\alpha_j=0$$

$$\rho(r)=\rho'(r)=0$$

$$\rho'''(r)=3!\sigma(r)$$

Where $\rho(r)$ *and* $\sigma(r)$ are first and second characteristics polynomials of our method. According to [13], the first condition is a sufficient condition for the associated block method to be consistent. Hence our method (2.13) is consistent.

3.3 Zero stability of the method

Definition 4.1 [14] the computational method is said to be zero-stable, if the roots $z_s,\ s=1,2,\cdots,k$ of the first characteristics polynomial $\rho(z)$ defined by $\rho(z)=\det\left(zA^{(0)}-E\right)$ satisfies $|z_s|\leq1$ and every root satisfies $|z_s|=1$ have multiplicity not exceeding the order of the differential equation. The first characteristic polynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right|=\left\|\begin{matrix}z&0&0&0&-1\\0&z&0&0&-1\\0&0&z&0&-1\\0&0&0&z&-1\\0&0&0&0&z-1\end{matrix}\right\|=z^4(z-1)$$

Thus, solving for z in

$$z^4(z-1) \tag{2.18}$$

Gives $z = 0, 0, 0, 0, 1$. Hence the block method (2.13) is said to be zero stable.

3.4 Convergence of the Block Method
Theorem 4.1: the necessary and sufficient conditions for linear multistep method to be convergent are that it must be consistent and zero-stable. Hence our method (2.13) formulated is consistent [13].

3.5 Region of Absolute Stability of our Method
Definition 4.2: the region of absolute stability is the region of the complex z plane, where $z = \lambda h$ for which the method is absolute stable. To determine the region of absolute stability of the block method, the methods that compare neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. Thus, the method according to [4] is called the boundary locus method. Applying this method we obtain the stability polynomial as

$$\bar{h}(w) = h^{15}\left(-\frac{1697}{5167968750\ \ 000000}w^4 - \frac{1}{2153320312\ \ 50000}w^5\right) + h^{12}\left(-\frac{2083381}{9187500000\ \ 00000}w^4 - \frac{144761}{7751953125\ \ 000000}w^5\right) + h^9\left(-\frac{235831}{7560000000\ \ 0}w^4 - \frac{1}{84000000}w^5\right) + h^6\left(-\frac{1}{26250000}w^4 - \frac{203113}{126000000}w^5\right) - \frac{13}{60}w^4h^3 + w^5 - \frac{5}{2}w^4 \tag{2.19}$$

Applying the stability polynomial in (2.19), we obtain the region of absolute stability in figure below.

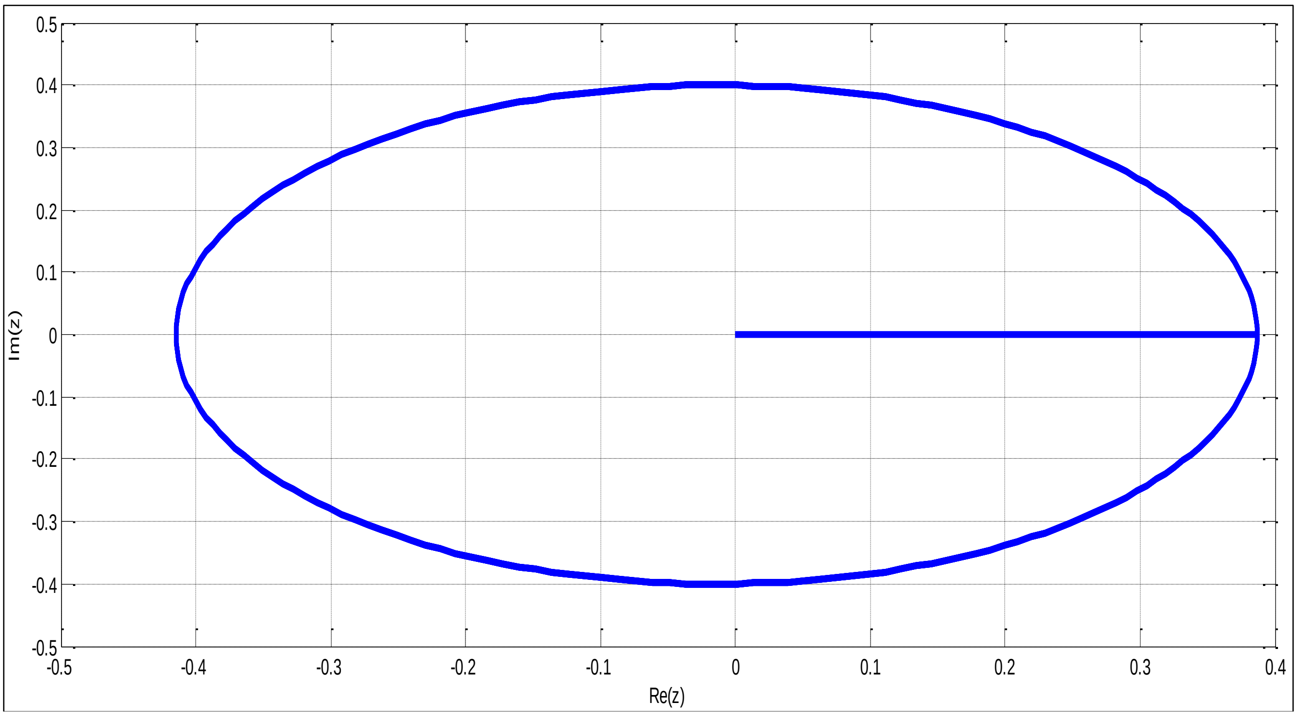

Fig 1: Region of absolute stability of our method

4. Numerical implementation of the method
In this section, we shall approximate some third order highly stiff and non-stiff initial value problems of the form (1.1), to test the efficiency and accuracy of our proposed method (2.13), (2.14) and (2.15) on three test problems. Our result are compared with the result obtain from the existing methods.

Problem 1: Consider the highly non-stiff third order problem
$y'''(x) = e^x, \quad y(0) = 3, y'(0) = 1, y''(0) = 5,$
with the exact solution: $y(x) = 2 + 2x^2 + e^x$

Source: [7, 2, 3]

Table 1: Showing the results for problem 1

X	Exact Result	Computed Result	Error in our Method	Error in [7]	Error in [2]	Error in [3]
0.1	3.12517091807564762480	3.30140275816017028360	6.87000e-17	0.0000e-00	2.66454e-15	6.34270e-13
0.2	3.30140275816016983390	3.30140275816017028360	4.49700e-16	2.8422e-13	4.44089e-16	2.32882e-12
0.3	3.52985880757600310400	3.52985880757600458560	1.48160e-15	1.6729e-12	3.10862e-15	5.44348e-12
0.4	3.81182469764127031780	3.81182469764127385700	3.53920e-15	2.9983e-11	6.66134e-15	9.85317e-12
0.5	4.14872127070012814680	4.14872127070013518300	7.03620e-15	3.1673e-11	9.76996e-15	1.59974e-11
0.6	4.54211880039050897490	4.54211880039052140450	1.24296e-14	9.1899e-11	2.04281e-14	2.37223e-11
0.7	4.99375270747047652160	4.99375270747049674680	2.02252e-14	8.9531e-11	2.13163e-14	3.35679e-11
0.8	5.50554092849246760460	5.50554092849249858620	3.09816e-14	1.9168e-10	1.86516e-14	4.53443e-11
0.9	6.07960311115694966380	6.07960311115699497970	4.53159e-14	2.1110e-10	2.22045e-14	5.97084e-11
1.0	6.71828182845904523540	6.71828182845910914590	6.39105e-14	4.9398e-10	2.13163e-14	7.64322e-11

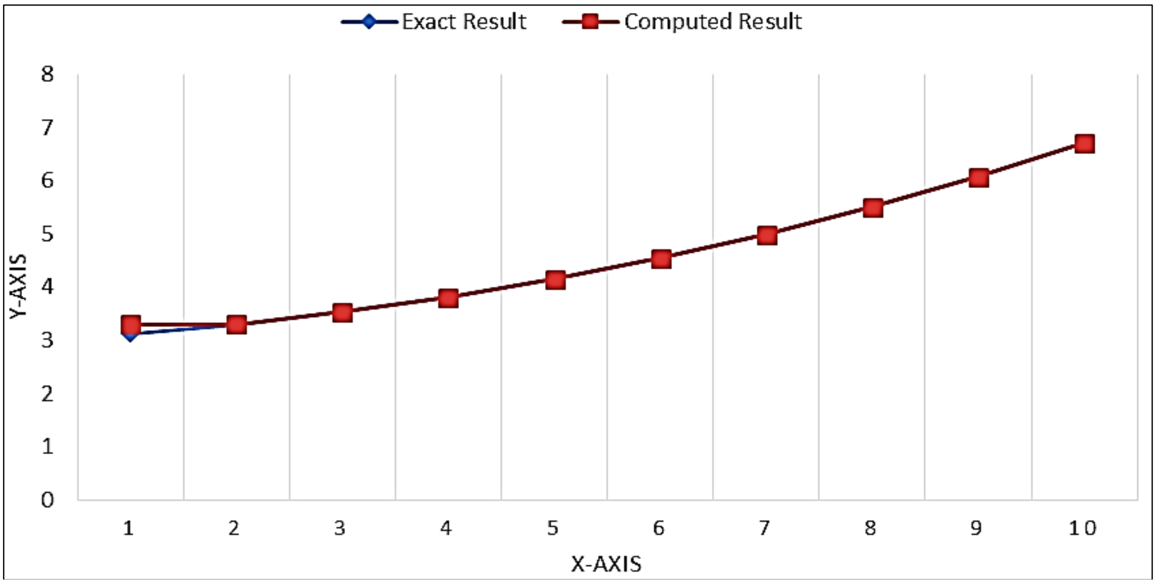

Fig 2: Graphical solution of problem 1

Problem 2: Consider the highly non-stiff third order problem

$$y'''(x) = 3\sin(x), \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,$$

with the exact solution: $y(x) = 3\cos(x) + \frac{x^2}{2} - 2$

Source: [2, 4, 3]

Table 1: Showing the results for problem 2

X	Exact Result	Computed Result	Error in our Method	Error in [2]	Error in [4]	Error in [3]
0.1	0.99001249583407729830	0.9900124958340772983	8.6100e-18	8.8818e-16	3.3307e-16	1.7282e-12
0.2	0.96019973352372489340	0.9601997335237248934	7.7950e-17	4.4409e-16	3.3307e-16	6.3179e-12
0.3	0.91100946737681805890	0.9110094673768180589	3.3814e-16	6.6613e-16	3.3307e-16	1.4295e-11
0.4	0.84318298200865524840	0.8431829820086552484	1.0049e-15	1.6653e-16	1.1102e-16	2.5020e-11
0.5	0.75774768567111814840	0.7577476856711181484	2.3769e-15	1.9984e-15	1.1102e-16	3.8928e-11
0.6	0.65600684472903489170	0.6560068447290348917	4.8329e-15	3.1086e-15	4.4409e-16	5.5360e-11
0.7	0.53952656185346527880	0.5395265618534652788	8.8285e-15	3.9968e-15	5.5511e-16	7.4644e-11
0.8	0.41012012804149626280	0.4101201280414962628	1.4891e-14	4.6074e-15	5.5511e-16	9.6128e-11
0.9	0.26982990481199336940	0.2698299048119933694	2.3614e-14	5.2181e-15	7.2164e-16	1.2002e-10
1.0	0.12090691760441915220	0.1209069176044191522	3.5652e-14	5.8703e-15	1.0547e-15	1.4570e-10

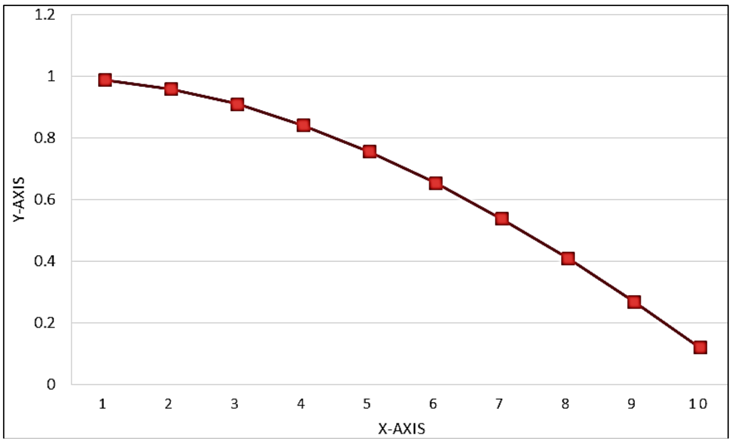

Fig 3: Graphical solution of problem 2

Problem 3: Consider the highly non-stiff third order problem

$$y'''(x)=3\cos(x), \quad y(0)=1, y'(0)=0, y''(0)=2,$$

With the exact solution: $y(x)=x^2-3\sin(x)+3x+1$

Source: Tarparki [20]

Table 3: Showing the results for problem 3

X	Exact Result	Computed Result	Error in our Method	Error in [20]
0.1	1.0104997500595155431O	1.0104997500595155431	1.9700e-16	2.4800e-07
0.2	1.04399200761481635360	1.0439920076148163536	1.2639e-15	7.3740e-06
0.3	1.10343938001598127470	1.1034393800159812747	4.0627e-15	6.0542e-05
0.4	1.19174497307404852500	1.1917449730740485250	9.4370e-15	2.5479e-04
0.5	1.31172338418739099920	1.3117233841873909992	1.8205e-14	7.7602e-04
0.6	1.46607257981489392840	1.4660725798148939284	3.1152e-14	1.9261e-03
0.7	1.65734693828692683900	1.6573469382869268390	4.9021e-14	4.1505e-03
0.8	1.88793172730143171510	1.8879317273014317151	7.2504e-14	8.3637e-03
0.9	2.16001927111754983460	2.1600192711175498346	1.0224e-13	1.4774e-02
1.0	2.47558704557631048000	2.4755870455763104800	1.3880e-13	2.4702e-02

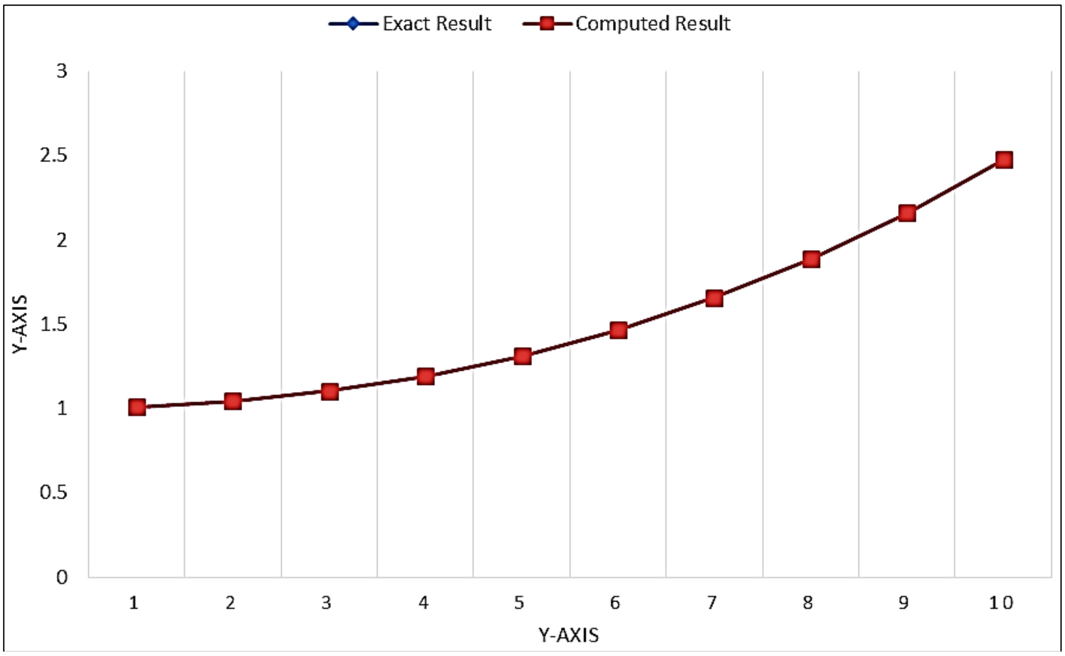

Fig 4: Graphical solution of problem 3

5. Conclusion

The numerical application of third derivative block hybrid method on third order initial value problem of ordinary differential equations is consider in this work. The method is derived by collocating and interpolating the approximate solution in power series, while Taylor series is used to generate the independent solution at selected grid and off grid points. The basic analysis of the method were established and it was found to be consistent, zero-stable and convergent. The developed method is then applied to solve some third order initial value problems of ODEs, and the result computed shows that the derived method is more accurate than some existing methods considered in this paper. We further plotted the solution graph of each problems and it is obvious that the numerical solution convergence toward the exact solution.

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