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A comprehensive study on establishing bilateral generating relations for certain hyper geometric functions of more than one variable

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Abstract
The present study focused on to establish some bilateral Generating relations for hyper geometric functions of two, three and four variables. Some particular cases have been explored.

Keywords: Hyper geometric, functions, variables namely

Introduction
The hypergeometric functions of four variables namely K (i = 9, 10, 11, 12, 13) were studied by Exton [2], while a study of hypergeometric functions of three variables can be found in Srivastava and Manocha [6]. Horn's functions, can be found in Erdélyi *et al.* [1]. In what follows, we mention following results, those are required in the present investigation.

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[-n,\mu,\upsilon,\alpha,x,y]t^n = (1-t)^{-\lambda} F_1\left[\lambda,\mu,\upsilon,\alpha,\frac{xt}{t-1},\frac{yt}{t-1}\right], \tag{1.1}$$

Where max $\left\{\left|\frac{xt}{t-1}\right|,\left|\frac{yt}{t-1}\right|,|t|\right\}<1.$ [cf. Srivastava] [5].

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_2[-n,\mu,\upsilon,\alpha,\beta,x,y]t^n = (1-t)^{-\lambda} F_2\left[\lambda,\mu,\upsilon,\alpha,\beta,\frac{xt}{t-1},\frac{yt}{t-1}\right], \tag{1.2}$$

Where max $\left\{\left|\frac{xt}{t-1}\right|,\left|\frac{yt}{t-1}\right|,|t|\right\}<1,$ [cf. Srivastava] [5].

$$\sum_{n=0}^{\infty} \binom{\upsilon+n}{n} \frac{(\lambda)_n}{(\mu)_n} P_{n+\upsilon}^{(\alpha,\beta-n-\upsilon)}(x)t^n = \binom{\alpha+\upsilon}{\upsilon}\left(\frac{1+x}{2}\right)^{\upsilon-\beta} F_2\left[\alpha+\upsilon+1,-\beta,\lambda,\alpha+1,\mu,\frac{1-x}{2},\frac{(x+1)t}{2}\right]. \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha,\beta-n)}(x)t^n = F_1\left[\lambda,-\beta,\lambda,\alpha+\beta+1,\mu,t,\frac{(x+1)t}{2}\right]. \tag{1.4}$$

Above two results are due to Srivastava [4].
Following four results are found in Srivastava and Manocha [6]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2F_1[\rho-n,\alpha,\beta,x]\, t^n = (1-t)^{-\lambda} F_1\left[\alpha,\rho,\lambda,\beta,x,-\frac{xt}{1-t}\right]. \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{(\delta)_n} 2F_1[-n,\alpha,\beta,x]\frac{t^n}{n!} = F_1[\gamma,\beta-\alpha,\alpha;\delta;t,(1-x)\,t]. \tag{1.6}$$

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(\lambda)_n}{(-\alpha-\beta-m)_n} P_{m+n}^{(\alpha-n,\beta-n)}(x)t^n = \binom{\alpha+\beta+2m}{m}\left(\frac{x+1}{2}\right)^m \left[1+\frac{(x+1)\,t}{2}\right]^{-\lambda}$$

$$\times F_1\left[-\beta-m,-m,\lambda;-\alpha-\beta-2m;\frac{2}{x+1},\frac{t}{1+(x+1)\,t/2}\right] \tag{1.7}$$

And

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2F_1[-n,a;b;x]\, t^n = (1-t)^{-\lambda}\, 2F_1\left[\lambda,a;b;-\frac{xt}{1-t}\right]. \tag{1.8}$$

In above results, whenever they appear, F_1 and F_2 are Appell's functions and ${}_2F_1(\cdot)$ is the Gauss hypergeometric function.

2. Main Results and Proofs

We have established the following bilateral generating relations:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_E[\lambda+n,\lambda+n,\lambda+n,\alpha,\beta,\beta;\gamma_1,\gamma_2,\gamma_3;x,y,z]\, F_1(-n,\mu,\upsilon,\gamma,u,v)\, t^n = (1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda)_p(\gamma)_p}{(\gamma_1)_p\, p!}\left(\frac{x}{1-t}\right)^p$$

$$\times K_9\left[\lambda+p,\lambda+p,\lambda+p,\lambda,p,\beta,\beta,\mu,\upsilon;\gamma_2,\gamma_3,\gamma,\gamma,\frac{y}{1-t},\frac{z}{1-t},\frac{ut}{t-1},\frac{vt}{t-1}\right] \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_F[\lambda+n,\lambda+n,\lambda+n,\alpha_1,\alpha_2,\alpha_1;\beta_1,\beta_2,\beta_2;x,y,z]\, F_2(-n,\mu,\upsilon;\alpha,\beta,u,v)\, t^n$$

$$=(1-t)^{-\lambda}\sum_{q=0}^{\infty}\frac{(\lambda)_q(\alpha_2)_q}{(\beta_2)_q\, q!}\left(\frac{y}{1-t}\right)^q$$

$$\times K_{10}\left[\lambda+q,\lambda,q,\lambda,q,\lambda+q,\alpha_1,\alpha_1,\mu,\upsilon,\beta_1,\beta_2+q,\alpha,\beta;\frac{x}{1-t},\frac{z}{1-t},\frac{ut}{t-1},\frac{vt}{t-1}\right]. \tag{2.2}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!} F_D^{(3)}(\lambda+n,\alpha_1,\alpha_2,\alpha_3;\gamma,x,y,z)\; 2F_1(-n,a;b;u)t^n$$

$$=(1-t)^{-\lambda} K_{11}\left[\lambda,\lambda,\lambda,\lambda,\alpha_1,\alpha_2,\alpha_3,a;\gamma,\gamma,\gamma,b;\frac{x}{1-t},\frac{z}{1-t},\frac{ut}{t-1}\right]. \tag{2.3}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!} F_G[\lambda,n,\lambda+n,\lambda+n,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z]\;\; P_n^{(\alpha,\beta-n)}(u)t^n$$

$$\sum_{p=0}^{\infty}\frac{(\lambda)_p(\beta_1)_p}{(\gamma_1)_p\, p!}\times p$$

$$\times K_{12}[\lambda+p,\lambda,p,\lambda,p,\lambda+p,\beta_2,\beta_2-\beta,\alpha+\beta+1;\gamma_2,\gamma_2,\mu,\mu;y,z,t,1/2(u+1)\,t]. \tag{2.4}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_G[\lambda,n,\lambda+n,\lambda+n,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z]\ F_2[-n,\mu,\upsilon,\alpha,\beta,u,v]\,t^n$$

$$=(1-t)^{-\gamma}\sum_{p=0}^{\infty}\frac{(\lambda)_p(\beta_1)_p}{(\gamma_1)_p\,p!}\left(\frac{x}{1-t}\right)^p$$

$$\times K_{13}\left[\lambda+p,\lambda+p,\lambda+p,\lambda+p,\beta_2,\beta_3,\mu,\upsilon,\gamma_2,\gamma_2,\alpha,\beta;\frac{y}{1-t},\frac{z}{1-t},\frac{ut}{t-1},\frac{vt}{t-1}\right]. \tag{2.5}$$

$$\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_n}{(\delta)_n\,n!}H_3[\alpha',\beta';\delta+n;x,y]\ {}_2F_1[-n,\alpha;\beta;u]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\alpha')_{2p}\times p}{(\delta)_p\,p!}$$

$$\times F_S[\alpha'+2p,\gamma,\gamma,\beta',\beta-\alpha,\alpha;\delta+p,\delta+p,\delta+p;y,t,(1-u)\,t] \tag{2.6}$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{(\lambda)_n}{(-\alpha-\beta-m)_n}H_3[\lambda+n;\gamma;\delta;x,y]\ P_{m+n}^{(\alpha-n,\beta-n)}(u)t^n$$

$$=\binom{\alpha+\beta+2m}{m}\left(\frac{u-1}{2}\right)^m\left[1+\frac{(u+1)\,t}{2}\right]^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda)_{2p}}{(\delta)_p\,p!}\left[\frac{x}{\left\{1+\frac{u+1}{2}\right\}^2}\right]^p$$

$$\times F_M\Big[\gamma,-\beta-m,-\beta-m,\lambda+2p,-m,\lambda+2p;\delta+p,-\alpha-\beta-2m,-\alpha-\beta-2m;$$

$$\left.\frac{y}{1+\frac{(u+1)\,t}{2}},\frac{u}{u+1},\frac{t}{1+\frac{(u+1)\,t}{2}}\right]. \tag{2.7}$$

$$\sum_{n=0}^{\infty}\binom{\upsilon+n}{n}\frac{(\lambda)_n}{n!}H_4[\lambda+n;\beta';\gamma,\delta;x,y]\,\mathrm{P}_{n+\upsilon}^{(\alpha,\beta-n-\upsilon)}(u)t^n$$

$$=\binom{\alpha+\upsilon}{\upsilon}\left(\frac{1+u}{2}\right)^{\upsilon-\beta}\sum_{p=0}^{\infty}\frac{(\lambda)_{2p}x^p}{(\gamma)_p P!}$$

$$\times F_K\left[\beta',\alpha+\upsilon+1,\alpha+\upsilon+1,\lambda+2p,-\beta,\lambda+2p;\delta,\alpha+1,\mu;y,\frac{1-u}{2},\frac{(1+u)\,t}{2}\right]. \tag{2.8}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}G_2[\lambda+n,\alpha',\beta,\beta';x,y]_2F_1(p-n,\alpha;\gamma,u)t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\alpha')_q (\beta)_q (-y)^q}{(1-\beta'-p)_q q!} \; F_M \left[\beta', \alpha, \alpha, \lambda, p, \lambda; 1-\beta-q, \gamma, \gamma, \frac{x}{t-1}, u, \frac{ut}{t-1} \right]. \quad (2.9)$$

And

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_4[\lambda+n, \beta; \gamma, -\delta; x, y] \, F_2(-n, \mu, \upsilon; \alpha, \beta; u, v) \, t^n$$

$$= (1-t)^{-\lambda} \sum_{q=0}^{\infty} \frac{(\lambda)_q (\beta)_q}{(\delta)_q q!} \left(\frac{y}{1-t} \right)^q X_8 \left[\lambda+q, \mu, \upsilon; \gamma, \alpha, \beta; \frac{x}{(1-t)^2}, \frac{ut}{t-1}, \frac{vt}{t-1} \right]. \quad (2.10)$$

Proofs of the Results

In the proofs of the above mentioned ten generating relations, we will use the relation

$$(\lambda)_n (\lambda+n)_{p+q+r} = (\lambda)_{n+p+q+r} = (\lambda)_{p+q+r} (\lambda+p+q+r)_n \quad (2.11)$$

Proof of (2.1)

Let us write A for the terms on the LHS of (2.1). Using (1.1) and (2.11), we have

$$A = \sum_{p,q,r=0}^{\infty} \frac{(\lambda)_{p+q+r} (\alpha)_p (\beta)_{q+r} x^p y^q z^r}{(\gamma_1)_p (\gamma_2)_q (\gamma_3)_r \, p! q! r!} (1-t)^{-(\lambda+p+q+r)}$$

$$F_1 \left(\lambda+p+q+r, \mu, \upsilon; r; \frac{ut}{t-1}, \frac{vt}{t-1} \right). \quad (2.12)$$

Now expressing $F_1(\cdot)$ in series and using (2.11), we get

$$A = (1-t)^{-\lambda} \sum_{p=0}^{\infty} \frac{(\lambda)_p (r)_p}{(r_1)_p \, p!} \left(\frac{x}{1-t} \right)^p \sum_{q,r,i,j=0}^{\infty} \frac{(\lambda+p)_{q+r+i+j} (\beta)_{q+r}}{(r_2)_q (r_3)_r (r)_{i+j}}$$

$$\times \frac{(\mu)_i (\upsilon)_j}{q! r! i! j!} \left(\frac{y}{1-t} \right)^q \left(\frac{z}{1-t} \right)^r \left(\frac{ut}{t-1} \right)^i \left(\frac{vt}{t-1} \right)^j. \quad (2.13)$$

By virtue of the expression for K_9 [cf. Exton [2, p.78]], the equation (2.13) yields the required result.
Employing (1.2), (1.4) and (1.8) the relations (2.2) - (2.5) will be obtained following above procedure.

Proof of (2.6)

Replace by B the terms on the LHS of (2.6) and then using (1.6) and (2.11), we get

$$B = \sum_{p,q=0}^{\infty} \frac{(\alpha')_{2p+q} (\beta')_q}{(\delta)_{p+q} \, p! q!} x^p y^q F_1(\gamma, \beta-\alpha, \alpha; \delta+p+q; t, (1-u)t) \quad .$$

Now write F_1 in terms of series and employ (2.11), we have

$$B = \sum_{p=0}^{\infty} \frac{(\alpha')_{2p}}{(\delta)_p \, p!} \times p \sum_{q,i,j=0}^{\infty} \frac{(\alpha'+2p)_q (r)_{i+j} (\beta')_q (\beta-\alpha)_i (\alpha)_j}{(\delta+p)_{q+i+j} \, q! i! j!} y^q t^i [(1-u)t]^j. \quad (2.14)$$

The result is obtained by using [6, p.68].
Results (2.7) – (2.10) are obtained in the manner (2.6) is obtained, we use (1.7), (1.3), (1.5) and (1.2) respectively.

3. Particular Cases

The particular cases, believed to be interesting, give bilateral generating relations for Appell's functions, for Laguerre polynomial and for the confluent form of F_2, ψ_1 and ψ_2 [see e.g. [6], p. 59].

(i) In (2.1), put x = 0, we then get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_4[\lambda+n,\beta,\gamma_2,\gamma_3;y,z]\, F_1(-n,\mu,\upsilon;\gamma,u,v)\, t^n$$

$$= (1-t)^{-\lambda} K_9\left[\lambda,\lambda,\lambda,\lambda,\beta,\beta,\mu,\upsilon,\gamma_2,\gamma_3,\gamma,\gamma,\frac{y}{1-t},\frac{z}{1-t},\frac{ut}{t-1},\frac{vt}{t-1}\right]. \tag{3.1}$$

(ii) Now, if in (2.1), $\upsilon = 0, \gamma_3 \to 0$ and if y is replaced by y / β and u by u / μ, then taking limit as $\mu, \beta \to \infty$, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \psi_1[\lambda+n,\alpha;\gamma_1,\gamma_2;x,y]\, {}_1F_1(-n;\gamma,u)\, t^n$$

$$= (1-t)^{-\lambda} \sum_{p=0}^{\infty} \frac{(\lambda)_p (\gamma)_p}{(\gamma_1)_p\, p!} \left(\frac{x}{1-t}\right)^p \psi_2\left[\lambda+p,\gamma_1,\gamma_2,\frac{ut}{t-1},\frac{y}{1-t}\right] \tag{3.2}$$

(iii) For $\beta_1 = 0$, (2.4) yields

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[\lambda+n,\beta_2,\beta_3,\gamma_2;y,z]\, P_n^{(\alpha,\beta-n)}(u) t^n$$

$$= K_{12}[\lambda,\lambda,\lambda,\lambda,\beta_2,\beta_2,-\beta,\alpha+\beta+1,\gamma_2,\gamma_2,\mu,\mu;y,z,t,1/2(u+1)\,t] \tag{3.3}$$

(iv). In (2.5) if we put $\beta_1 = \beta_3 = \upsilon = 0$ and replace u by u / μ, by taking limit as $\mu \to \infty$ and putting $\alpha = \lambda$, we get

$$\sum_{n=0}^{\infty} {}_2F_1(\lambda+n,\beta_2;\gamma_2;y) L_n^{(\lambda-1)}(u) t^n = (1-t)^{-\lambda} \psi_1\left[\lambda,\beta_2;\gamma_2,\lambda;\frac{y}{1-t},\frac{ut}{t-1}\right]. \tag{3.4}$$

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