International Journal of Statistics and Applied Mathematics 2020; 5(2): 09-16

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2020; 5(2): 09-16

www.mathsjournal.com
Received: 03-01-2020
Accepted: 05-02-2020

Kaze Atsi
Mathematics Department, Federal University, Gashua, P.M.B. 1005. Yobe State, Nigeria

GM Kumleng
Mathematics Department, Federal University, Gashua, P.M.B. 1005. Yobe State, Nigeria

Corresponding Author:
Kaze Atsi
Mathematics Department, Federal University, Gashua, P.M.B. 1005. Yobe State, Nigeria

A family of modified backward differentiation formula (BDF) type block methods for the solution of stiff ordinary differential equations

Kaze Atsi and GM Kumleng

Abstract
A family of modified BDF type block methods for the solution stiff ordinary differential equations has been constructed in this research. Four different block methods for step number, k =2, 3, 4 and 5 have been derived using the multi-step collocation approach. The stability properties of the newly constructed methods have been investigated using written computer codes and have all shown to be convergence and A-stable. The new methods are found to be suitable for computing solutions of stiff problems. The solutions of the new block methods have been compared with the corresponding exact solutions and the associated absolute errors are presented. Performance of the new methods improved as the step number increase. The implementation approach adopted, contributed both in the accuracy and in managing computation effort.

Keywords: BDF, block methods, stiff ordinary differential equations (SODEs), stability, consistency

1. Introduction

Real life problems both in physical and social sciences, and almost every field of study where measurement can be taken, are more clearly express through mathematical models. These models take the form of differential equations. Generally, differential equations are of different types and exhibit different phenomenon.

Stiff differential equation is a term used for a special class of differential equation which exhibit 'stiffness' phenomenon. The initial value problem with stiff differential equation is most notable in engineering sciences. It is also present in control systems and many non-industrial areas like weather prediction and biology. A differential equation is stiff when no eigenvalue of the jacobian matrix has a real part which is at all large and positive.

Interestingly, the exact solutions of some of the Initial Value Problems (IVPs) of Stiff Ordinary Differential Equations (SODEs) of the form:

$$y' = f(x, y), y(x_0) = y_0, x \in [a, b], y \in R \tag{1.1}$$

Do not exist because of the difficulty in solving them. Hence, suitable numerical methods are developed for solving this kind of differential equation. The first numerical methods to have been proposed for solving stiff differential equations were the Backward Differentiation Formulae (BDF) methods, Curtiss and Hirschfelder (1952). In this thesis, a family of modified BDF type block methods is developed for solving (1.1).

2. Materials and Methods

Based on the interpolation and collocation methods, we consider the discrete form of our new method given by:

$$\sum_{j=0}^{k-1} \alpha(x)_j y_{n+j} = h\left(f_{n+k}\beta(x)_k + f_{n-1+k}\beta(x)_{k-1}\right) \tag{2.1}$$

Where $\alpha_j(x)$ and $\beta_j(x)$ are the continuous coefficients of the method, *h* is the step size and the distinct interpolation points are the same as *k*.

Generating the interpolation and collocation points of equation (2.1) at $x = x_n, x_{n+1}, \ldots, x_{n+k-1}$ and at $x = x_{n+k-1}, x_{n+k}$ respectively, yield the following D matrix:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{(k+1)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & (x_n + (k-1)h) & (x_n + (k-1)h)^2 & \cdots & (x_n + (k-1)h)^{k+1} \\ 0 & 1 & 2(x_n + (k-1)h) & \cdots & (x_n + (k-1)h)^k \\ 0 & 1 & 2(x_n + kh) & \cdots & (k+1)(x_n + kh)^k \end{bmatrix}$$

Using the maple software package gives the inverse, D^{-1}

$$C = \begin{bmatrix} \alpha_{0,0} & \cdots & \alpha_{k-1,0} & \beta_{k-1,0} & \beta_{k,0} \\ \alpha_{0,1} & \cdots & \alpha_{k-1,1} & \beta_{k-1,1} & \beta_{k,1} \\ \alpha_{0,2} & \cdots & \alpha_{k-1,2} & \beta_{k-1,2} & \beta_{k,2} \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \alpha_{0,k+1} & \cdots & \alpha_{k-1,k+1} & \beta_{k-1,k+1} & \beta_{k,k+1} \end{bmatrix}$$

The columns of $C = D^{-1}$ in turn, are then used to generate the values of the continuous coefficients,

$$\alpha_{0,j}(x), \ldots, \alpha_{k-1,j}(x), \beta_{k-1,j}(x), \beta_{k,j}(x); j = 0, 1, \ldots, k+1 \tag{2.2}$$

The continuous form of *k*-step block methods will constitutes the values of the continuous coefficients above, after some manipulations,

$$y(x) = \alpha_0(x)\, y_n + \alpha_1(x)\, y_{n+1} + \ldots + \alpha_{k-1}(x)\, y_{k-1} + \beta_{k-1}(x) f_{n+k-1} + \beta_k(x) f_{n+k} \tag{2.3}$$

Where $y_{n+j} = y(n+j)$ and $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right)$

$y(x)$ is then evaluated at point $x = x_{n+k}$ and its derivative $y'(x)$ at points $x = x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-2}$, yielding *k* discrete members.

To implement (2.3), the approach adopted in Chollom et al. (2007) and Sunday et al. (2013) has been followed. This gives a continuous discrete block scheme of the form,

$$AY_m = By_m + hbf_m + hCF_m \tag{2.4}$$

Where,

$Y_m = \left[y_{n+1}, y_{n+2}, \ldots, y_{n+k}\right]^T$; $y_m = \left[y_{n-k}, \ldots, y_{n-1}, y_n\right]^T$; $f_m = \left[f_{n-k}, \ldots, f_{n-1}, f_n\right]^T$ and $F_m = \left[f_{n+1}, f_{n+2}, \ldots, f_{n+k}\right]^T$

B, *b*, *C* are *k* by *k* matrices and *A* is a *k* by *k* identity matrix

For the computational approach, the jacobian of the *k*-step block methods are generated and incorporated into a MATLAB written code. The specifications of the new block methods are found in the subsequent sections.

2.1 Specifications of the new block methods
Evaluating the continuous scheme of equation (2.3) at some selected point and its first derivative w.r.t x at $x = x_0$ yield the following schemes:
The new block methods for $k = 2$ yield the following discrete formula

$$y_{n+1} - y_n = \frac{1}{12} h\left(8f_{n+1} - f_{n+2} + 5f_n\right) \tag{2.5}$$

The new block methods for $k = 3$ yield the following discrete formula

$$y_{n+2} - \frac{32}{19} y_{n+1} + \frac{13}{19} y_n = \frac{h}{19}\left(-\frac{17}{3} f_n + 13f_{n+2} - \frac{4}{3} f_{n+3}\right) \tag{2.6}$$
$$y_{n+2} - \frac{8}{9} y_{n+1} - \frac{1}{9} y_n = \frac{h}{27}\left(17f_{n+1} + 14 f_{n+2} - f_{n+3}\right)$$

The new block methods for $k = 4$ yield the following discrete formula

$$y_{n+3} - \frac{459}{268} y_{n+2} + \frac{81}{67} y_{n+1} - \frac{133}{268} y_n = \frac{3}{67} h\left(\frac{37}{8} f_n + 14f_{n+3} - \frac{9}{8} f_{n+4}\right) \tag{2.7}$$

$$y_{n+3} - \frac{315}{152} y_{n+2} + \frac{18}{19} y_{n+1} + \frac{1}{8} y_n = \frac{3h}{152}\left(-37f_{n+1} + 29f_{n+3} - 2hf_{n+4}\right)$$

$$y_{n+3} - \frac{306}{413} y_{n+2} - \frac{117}{413} y_{n+1} + \frac{10}{413} y_n = \frac{3}{413} h\left(111f_{n+2} + 62f_{n+3} - 9f_{n+4}\right)$$

The new block methods for $k = 5$ yield the following discrete formula

$$y_{n+4} - \frac{179}{97} y_{n+3} + \frac{567}{388} y_{n+2} - y_{n+1} + \frac{149}{388} y_n = \frac{1}{3880} h\left(2235\, hf_{n+4} - 144\, hf_{n+5} - 591 f_n\right) \tag{2.8}$$

$$y_{n+4} - \frac{117}{59} y_{n+3} + \frac{864}{413} y_{n+2} - y_{n+1} - \frac{45}{413} y_n = \frac{h}{826}\left(591 f_{n+1} + 456 f_{n+4} - 27 f_{n+5}\right)$$

$$y_{n+4} - \frac{8976}{3703} y_{n+3} + \frac{540}{529} y_{n+2} + \frac{1616}{3703} y_{n+1} - \frac{123}{3703} y_n = \frac{h}{3703}\left(-4728f_{n+2} + 1884f_{n+4} - 96f_{n+5}\right)$$

$$y_{n+4} - \frac{2792}{4895} y_{n+3} - \frac{2484}{4895} y_{n+2} + \frac{424}{4895} y_{n+1} - \frac{43}{4895} y_n = \frac{h}{4895}\left(4728f_{n+3} + 2004\, f_{n+4} - 72f_{n+5}\right)$$

3. Results and Discussions
3.1 Analysis of Block Methods
The orders and error constants of the new block methods are obtained using Chollom *et al*. (2007). Hence,

The block method for k=2 has order $p = (3, 3)^T$ with error constant $\overline{C}_{p+1} = \left(\frac{1}{24}, 0\right)^T$

The block method for k=3 has order $p = (4, 4, 4)^T$ with error constant $\overline{C}_{p+1} = \left(-\frac{19}{720}, -\frac{1}{90}, -\frac{3}{80}\right)^T$

The block method for $k = 4$ has order $p = (5, 5, 5, 5)^T$ with error constant $\overline{C}_{p+1} = \left(\frac{3}{160}, \frac{1}{90}, \frac{3}{160}, 0\right)^T$

The block method for k=5 has order $p = (6, 6, 6, 6, 6)^T$ with error constant

$$\overline{C}_{p+1} = \left(-\frac{863}{60480}, -\frac{37}{3780}, -\frac{29}{2240}, -\frac{8}{945}, -\frac{275}{12096}\right)^T$$

3.2 Definition: The sufficient condition for the associated block method to be consistent is that the order $p \geq 1$**.** By this definition, therefore, the new block methods are all consistent.

3.3. Definition: A block method is said to be zero stable if the roots r_s , s = 1, n of the first characteristic polynomial $\overline{\rho}(r)$, defined by

$$\overline{\rho}(r) = det[rA - B] \tag{3.1}$$

satisfies $|r_s| \leq 1$ and every root with $|r_s| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$

3.4. Zero-stability of the new block methods

For k=2, and as $h \to 0$ we simplify

$$det\left[r\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}\frac{2}{3} & -\frac{1}{12}\\ \frac{4}{3} & \frac{1}{3}\end{bmatrix}z\right) - \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\right] = 0$$

$$det\begin{bmatrix} r\left(1 - \frac{2}{3}z\right) & \frac{1}{12}rz - 1 \\ -\frac{4}{3}rz & r\left(-\frac{1}{3}z + 1\right) - 1 \end{bmatrix} = 0$$

This implies that

$$r^2 - r^2 z - r + \frac{1}{3} r^2 z^2 - \frac{2}{3} rz = 0 \tag{3.2}$$

Solving for r, we have

$$\left[[r=0], \left[r = \frac{2z+3}{z^2 - 3z + 3}\right]\right]$$

Evaluating at point z=0, we have $r = 1$

Therefore, $[r=0]$ or [r=1]. Hence, the new block method for k=2 is zero-stable.

Following the same definition, the block methods for k=3, 4 and 5 are found to also be zero-stable.

3.5 Definition: A numerical method is said to be convergent if it is both consistent and zero stable (**Lambert, 1973**).

By the above definition, therefore, the new constructed methods are found to be convergent.

3.6 Absolute stability region of the new block methods:

Following Chollom *et al*. (2007) [4] and Butcher (1985), the new block methods are reformulated as General linear methods and the region of absolute stability of the method was plotted using the MATLAB program and are shown in Fig. 1-4. Figure 1-4 reveal that the new block methods are A-stable.

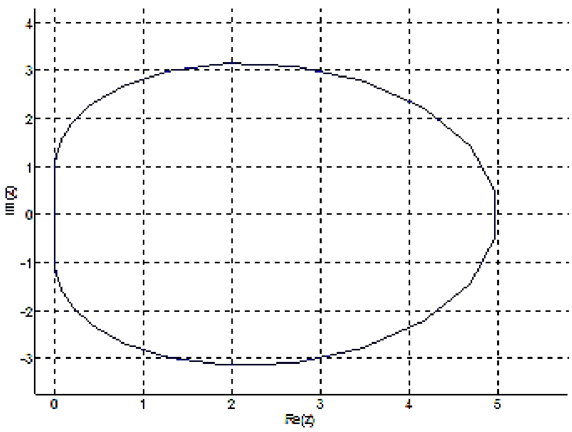

Fig 1: Absolute Stability region for k=2

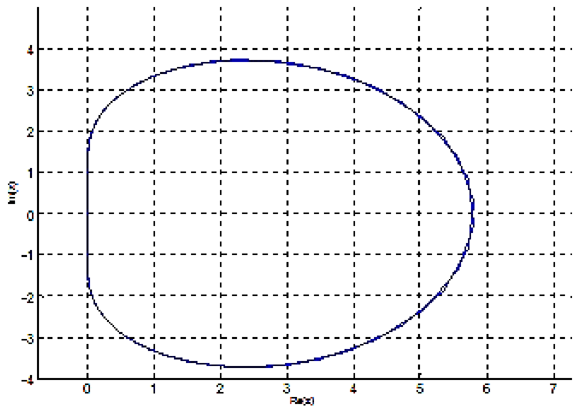

Fig 2: Absolute Stability region for k=3

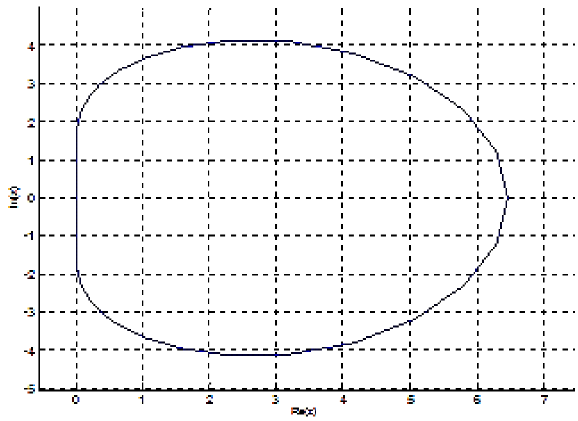

Fig 3: Absolute Stability region for k=4

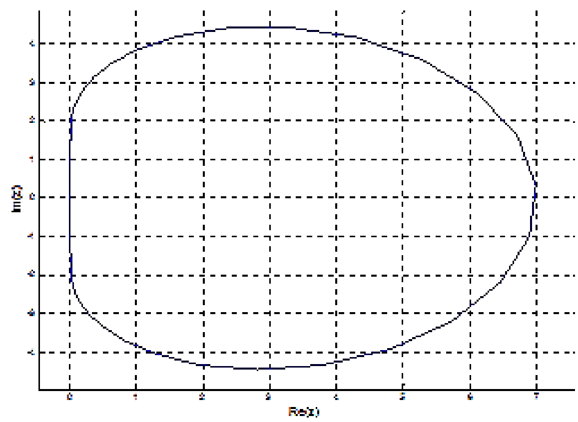

Fig 4: Absolute Stability region for k=5

3.7 Numerical experiments
The numerical experiments are carried out by incorporating the jacobian of the k-step block methods into a MATLAB written code. The absolute errors calculated in the code are defined as

$$Erc = |yc - yex| \tag{3.3}$$

Where yex is the exact solution, yc is the computed solution and Erc is the absolute error.

3.8 Numerical examples
For each new class of block methods, three problems are solved to test their efficiency with constant step size, $h=0.1$ in the range $0 \leq t \leq 10$.

Example 1: This problem is taken from Kumleng et al. (2013) [6]

$$y'_1 = 998 y_1 + 1998 y_2 \qquad y_1(0) = 1$$

$$y'_2 = - 999 y_1 - 1999 y_2 \qquad y_2(0) = 1$$

And the exact solution

$$y_1 = 4e^{-t} - 3e^{-1000t}$$

$$y_2 = -2e^{-t} + 3e^{-1000t}$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -1000$

Example 2: this is a physical problem taken from Musa et al. (2013)

$$y'_1 = -21y_1 + 19y_2 - 20y_3 \qquad y_1(0) = 1$$

$$y'_2 = 19y_1 - 21y_2 + 20y_3 \qquad y_2(0) = 0$$

$$y'_3 = 40y_1 - 04y_2 - 40y_3 \qquad y_3(0) = -1$$

The exact solution is as follows

$$y_1(x) = 0.5[\, e^{-2x} + e^{-40x}(cos40x + sin40x)\,]$$

$$y_2(x) = 0.5[\, e^{-2x} - e^{-40x}(cos40x + sin40x)\,]$$

$$y_3(x) = 2e^{-40x}\left(-\frac{1}{2}\,cos40x + \frac{1}{2}\,sin40x\right)$$

The eigenvalues are -2, -40 + 40i, and -40 -40i.
The following solution curves are obtained for the new block method, k=2

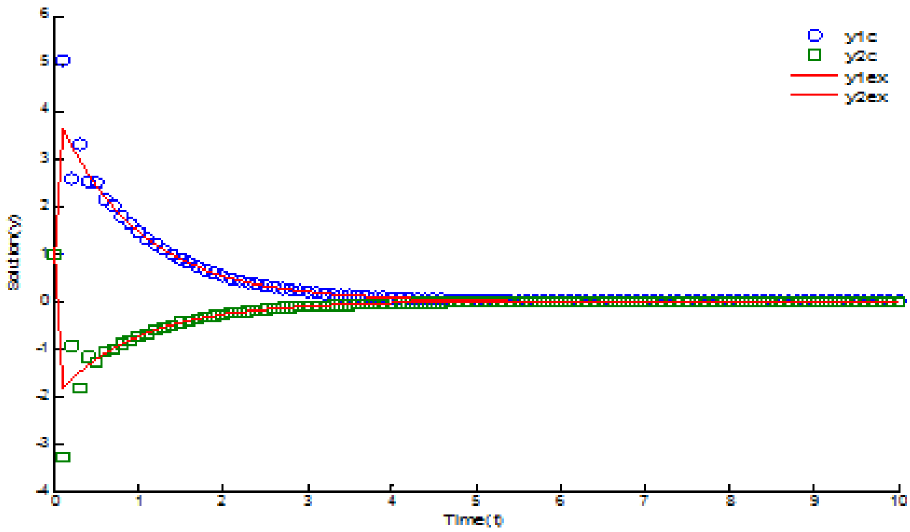

Fig 1: NCBM2 and Exact solutions of Example 1

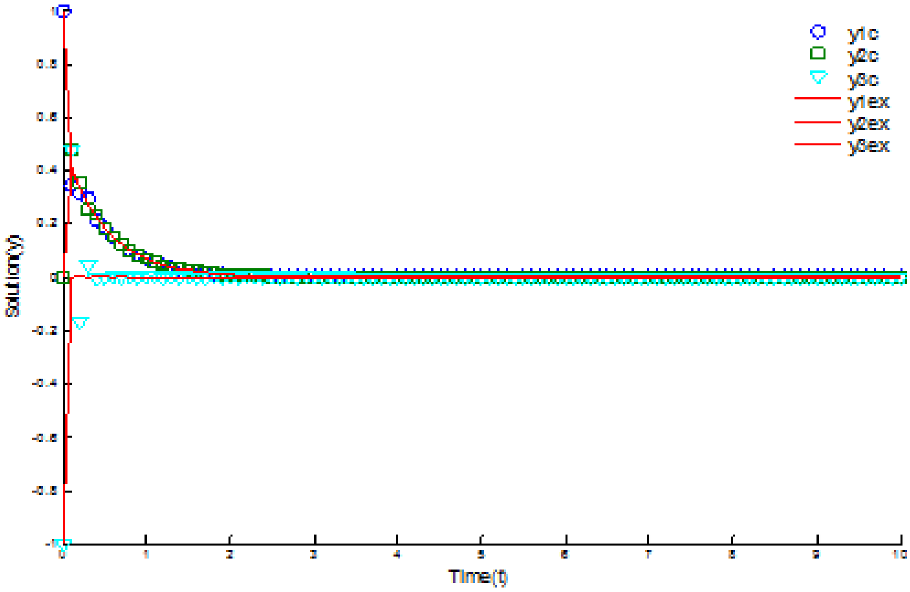

Fig 3: NCBM2 and Exact solutions of Example 2

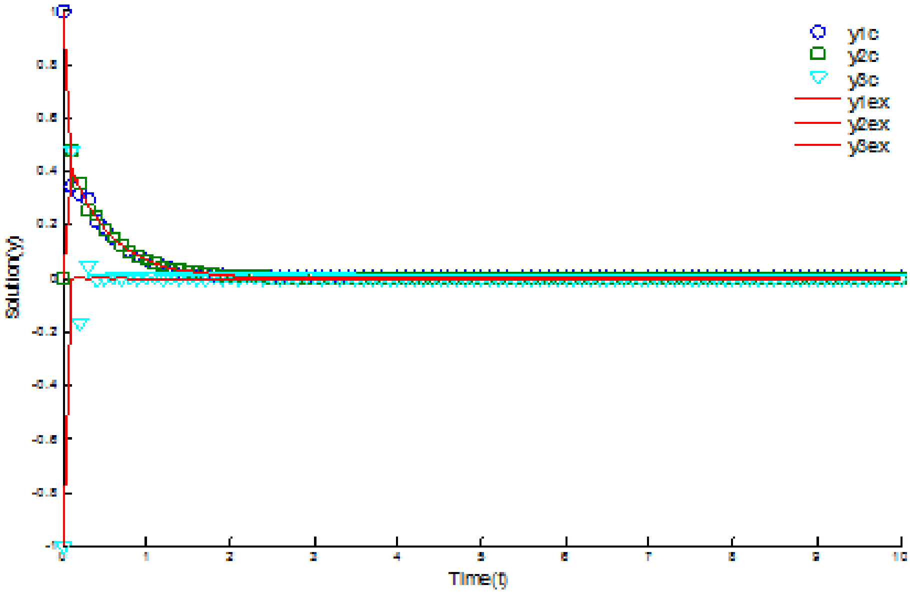

Fig 3: NCBM2 and Exact solutions of Example 3

The same procedure are used to obtain the solution curves for k=3, 4 and 5 of experimental examples.

4. Conclusion

The new methods are implemented as self-starting on three different systems of stiff ordinary differential equations collected from various literatures. The performance of the methods improved as the step number increase. Accuracy and efficiency of a method is dependent on the implementation strategies. Therefore, the implementation approach adopted, contributed immensely in only the accuracy, but also in managing computation effort. The authors in the next work will address the construction block methods with higher step number.

References

1. Ajie IJ, Ikhile MNO, Onumanyi P. A family of block methods derived from tom and bdf pairs for stiff ordinary differential equations. American Journal of Mathematics and Statistics. 2014; 4(2):121-130. Doi: 10.5923/j.ajms.20140402.08.
2. Akinfenwa OA, Jator SN, Yao NM. On the stability of continuous block backward differentiation formula for solving stiff ordinary differential equations. Modern Science Publishers. 2012; 3(2):50-58.
3. Althemai JM, Skwame Y, Donald JZ. Multiple off-grid hybrid block Simpson's methods for solution of stiff ordinary differential equations. International Journal of Science and Research. 2016; 5(2):1104-1109.
4. Chollom JP, Ndam JN, Kumleng GM. On some properties of the block linear multi-step methods. Science World Journal. 2007; 2(3):11-17.
5. Ehigie JO, Okunuga SA. *L(α)*-stable second derivative block multistep formula for stiff initial value problems. IAENG International Journal of Applied Mathematics, 2014, 44(3).
6. Kumleng GM, Adee SO, Skwame Y. Implicit two step adams moulton hybrid block method with two off-step points for solving stiff ordinary differential equations. Journal of natural sciences research. 2013; 3(9):77-81.

7. Sunday J, Odekunle MR, Adesanya AO, James AA. Extended block integrator for first-order stiff and oscillatory differential equations. American Journal of Computational and Applied Mathematics. 2013; 3(6):283-290. Doi:10.5923/j.ajcam.20130306.03.
8. Thohura S, Rahman A. Numerical Approach for Solving Stiff Differential Equations: A Comparative Study. Global journal of science frontier research, 2013, 13(6).