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On unitary equivalence of some classes of operators in Hilbert spaces

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Abstract
It is a well-known result in operator theory that whenever two operators are similar then they have equal spectra even though they do not have to belong to the same class of operators. However under a stronger relation of unitary equivalence it can be shown that two unitarily equivalent operators may belong to the same class of operators. In this paper we endeavor to exhibit results on such classes of operators which belong to same class under unitary equivalence.

Keywords: Binormal operator, unitary equivalence

1. Introduction

Let H be a complex Hilbert space and $B(H)$ denote the Banach algebra of all bounded linear operators on H. An operator $X \in B(H)$ is said to be a quasi-affinity if X is both one to one and has dense range. We also have that two operators A and B are said to be similar if there exists an invertible operator S such that $A = S^{-1}BS$. If there exists a unitary operator U such that $A = U^*BU$ then A and B are said to be unitarily equivalent. While A and B are said to be quasi-similar if there exists quasi-affinities X and Y such that $AX = XB$ and $BY = YA$. These properties of similarity, unitary equivalence and quasi-similarity have been studied by a number of authors who by a large extent relate them to equality of spectra for such operators. However no adequate investigation of unitarily or isometric equivalent operators belonging to the same class of operators has been addressed. R.G. Douglas [2] showed that quasi-similar normal operators are unitarily equivalent and recently B.M Nzimbi, G.P, Porkariyal and S.K. Moindi [5] showed that if a normal operator T is unitarily equivalent to another operator S then S is also normal. In this paper we show that a pair of unitarily equivalent operators may belong to the same class of operators of which some are larger than the class of normal operators.

2. Notations, definitions and terminologies

Given a complex Hilbert space H and operators $A, B \in B(H)$ then commutator of A and B is given by $[A,B] = AB - BA$. The spectrum of A is denoted by $\sigma(A) = \{\lambda \in \mathbb{C}: A - \lambda I isnotinvertible\}$ where $\mathbb{C}$ is the complex number field. We denote range of A and kernel of A by ranA and kerA respectively. Recall that an operator $A \in B(H)$ is said to be:

Binormal if $[A^*A, AA^*] = 0$
Quasinormal if $[A^*, A^*A] = 0$
Normal if $[A^*, A] = 0$
Hyponormal if $A^*A \geq AA^*$
Partial isometry if $A = AA^*A$
Isometry if $A^*A = I$
Coisometry if $AA^* = I$
Unitary if $A^*A = AA^* = I$

We also have the following inclusion of classes of operators which are key to prooving some of the results to follow.

$Normal \subseteq Quasinormal \subseteq Binormal$
$Normal \subseteq Subnormal \subseteq Hyponormal$
$Unitary \subseteq Isometry \subseteq PartialIsometry$
$Unitary \subseteq Coisometry \subseteq PartialIsometry$

3. Main results

Theorem 3.1
Let $A, B \in B(H)$ be such that A is binormal and $A = UBU^*$ where U is an isometry. Then B is also binormal.

Proof
Since A is binormal we have that $[A^*A, AA^*] = 0$. We also have $A = UBU^*$ implies that $A^* = UB^*U^*$. Thus $U^*A = BU^*$ and $U^*A^* = B^*U^*$. That is
$A^*A = UB^*U^*UBU^*$
$= UB^*BU^*$
and
$AA^* = UBU^*UB^*U$
$= UBB^*U^*$
But $[A^*A, AA^*] = 0$ implies that $UB^*BU^*UBB^*U^* = UBB^*U^*UB^*BU^*$ which simplifies to

$UB^*BBB^*U^* = UBB^*B^*BU^* \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Pre-multiplying equation 1 by U^* and again post-multiplying it by U we obtain the result
$U^*UB^*BBB^*U^*U = U^*UBB^*B^*BU^*U$. Which simplifies to $B^*BBB^* = BB^*B^*B$ implying $[B^*B, BB^*] = 0$. Hence B is binormal.

Remark 3.2
We note from theorem 3.1 that binormal operators are unitarily invariant

Corollary 3.3
Let A and B be such that $A = UBU^*$ where U is an isometry. If A belongs to any of the following classes of operators then B also belongs:
1. Normal
2. Quasinormal

Proof
It suffices to note that both normal and quasi-normal are subclasses of binormal operators.

Remark 3.4
B.M. Nzimbi *et al.* [5] proved the following result which becomes a consequence of corollary 3.3 as stated above.

Theorem A [5, Theorem 2.1]
If T is a normal operator and S is unitarily equivalent to T then S is also normal.

Theorem 3.5
Let $A, B \in B(H)$ be such that A is binormal and $A = U^*BU$ where U is a Co-isometry. Then B is also binormal.

Proof
Since $A = U^*BU$ we have that $A^* = U^*B^*U$
$A^*A = U^*B^*UU^*BU = U^*B^*BU$
$AA^* = U^*BUU^*B^*U = U^*BB^*U$
$A^*AAA^* = U^*B^*BUU^*BB^*U$
$= U^*B^*BBB^*U$
And
$AA^*A^*A = U^*BB^*UU^*B^*BU$
$= U^*BB^*B^*BU$
i.e. $U^*B^*BBB^*U = U^*BB^*B^*BU$
i.e. $B^*BBB^* = BB^*B^*B$
i.e. $[B^*B, BB^*] = 0$
Hence B is also binormal.

Remark 3.6
In view of both theorems 3.1 and 3.2 above We can also assert that binormal operators are both isometrically and co-isometrically invariant.

Theorem 3.7
Let A be a hyponormal operator and B be another operator such that either
1. $A = UBU^*$ where U is an isometry or
2. $A = U^*BU$ where U is a co-isometry.

Then B is also hyponormal

Proof
1. $A = UBU^*$ implies $A^* = UB^*U^*$
$\therefore A^*A = UB^*U^*UBU^*$
$= UB^*BU^*$
And
$AA^* = UBU^*UB^*U^*$
$= UBB^*U^*$

Note that $A^*A \geq AA^*$ implies

$UB^*BU^* \geq UBB^*U^* \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Pre-multiplying relation 2 by U^* and also post-multiplying each side of 2 by U gives
$B^*B \geq BB^*$
Hence B is hyponormal.
1. $A = U^*BU$ implies $A^* = U^*B^*U$
$\therefore A^*A = U^*B^*UU^*BU$
$= U^*B^*BU$
And
$AA^* = U^*BUU^*B^*U$
$= U^*BB^*U$
Note that $A^*A \geq AA^*$ implies

$U^*B^*BU \geq U^*BB^*U \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Pre-multiplying relation 3 by U and also post-multiplying each side of (3) by U^* gives
$B^*B \geq BB^*$
Hence B is hyponormal.
The following corollary is immediate as a result of theorem 3.7.

Corollary 3.8
Hyponormal operators and in particular subnormal operators are unitarily invariantnt.
The following result shows that partial isometries are also isometrically and co-isometrically invariant.

Theorem 3.9
Let A be a partial isometry and B be any other operator such that either
1. $A = UBU^*$ where U is an isometry or
2. $A = U^*BU$ where U is a co-isometry

Then B is also a partial isometry.

Proof

1. We note that $A = UBU^*$ implies that $A^* = UB^*U^*$. Since A is a partial isometry we have $A = AA^*A$. Thus

$A = AA^*A$
$= UBU^*UB^*U^*UBU^*$
$= UBB^*BU^*$
$= UBU^*$

i.e. $UBB^*BU^* = UBU^*$…………………………………………..(4)

Pre-multiplying equation 4 by U^* and then post-multiplying it by U we obtain
$BB^*B = B$
Hence B is also a partial isometry.

1. Similarly given $A = U^*BU$ implies $A^* = U^*B^*U$, and A being a partial isometry we have that $A = AA^*A$. Thus

$A = AA^*A$
$= U^*BUU^*B^*UU^*BU$
$= U^*BB^*BU$
$= U^*BU$

i.e. $U^*BB^*BU = U^*BU$……………………………………….(5)

Pre-multiplying equation 5 by U and then post-multiplying it by U^* We obtain
$BB^*B = B$
Hence B is also a partial isometry.

4. References

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