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Recurrence relations for the generalized hypergeometric polynomial set S_n (x, y)

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Abstract
In the present paper an attempt has been made to express some recurrence relations for the generalized hypergeometric polynomial set Sn (x, y) followed by important and interesting particular cases. Out of these particular results some of them stand for well known polynomials and some of them are believed to be new. Those recurrence relations are of at most important for mathematicians, scientists and engineers.

Keywords: Hypergeometric polynomial, orthogonal polynomial, generating function

1. Introduction

We have recently defined the generalized hypergeometric polynomial set S_n (x, y) by means of the generating functions [1],

$$e^{\lambda yt}\, F\left[\begin{matrix}(G_r);\\ \lambda_1 y^{e_1} t^{e_1}\\ (H_s);\end{matrix}\right] \times F\left[\begin{matrix}(a_p);(A_h);(C_u)\\ \lambda_3 x^{e_3} t, \lambda_2 x^{e_2} y^{-e_2} t^{e_2}\\ (b_q);(B_k);(D_v)\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} S_{n,e_1;e_2;e_3;(H_s);(b_q);(B_k);(D_v)}^{\lambda;\lambda_1;\lambda_2;\lambda_3;(G_r);(a_p);(A_h);(C_u)}(x,y)\, t^n \qquad \ldots (1.1)$$

Where $\lambda, \lambda_1, \lambda_2, \lambda_3$ are real and e_1, e_2, e_3 are positive integers.

The left hand side of (1.1) contains Appell function of two variables in the notation of Burchnall and Chaundy[2]. The polynomial set contains a number of parameters, for simplicity, we shall denote.

$$S_{n,e;e_1;e_2;e_3;(H_s)(b_q);(B_k);(D_v)}^{\lambda;\lambda_1;\lambda_2;\lambda_3;(G_r);(a_p);(A_h);(C_u)}(x,y)$$

by S_n (x, y).

Where n denote the order of the polynomial set.
After little simplification (1.1) gives

$$S_n(x,y) = \sum_{\substack{m,m_1,m_2>0\\ m+e_1m_1+e_2m_2\le 0}} \frac{\Delta(m_1,m_2)}{(n-m-e_1m_1-e_2m_2)!}$$

Where $\Delta(m_1,m_2)=\dfrac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$

$$S_n(x,y)=\sum_{\substack{m,m_1,m_2>0\\ m+e_1m_1+e_2m_2\le 0}}\frac{\Delta(m_1,m_2)}{(n-m-e_1m_1-e_2m_2)!} \quad \text{… (1.2)}$$

Where $\Delta(m_1,m_2)=\dfrac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$

$$\times\frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_v)\right]_{m_2}}$$

$$\times\frac{x^{m_2e_2}\lambda^m\lambda_1^{m_1}\lambda_2^{m_2}\left(\lambda_3x^{e_3}\right)^{n-m-e_1m_1-e_2m_2}\;y^{m+e_1m_1+e_2m_2}}{m!\quad m_1!\quad m_2!\;\;(n-m-e_1m_1-e_2m_2)!}$$

The polynomial set S_n (x, y) happens to the generalization of as many as forty-one orthogonal and non- orthogonal polynomials.

2. Notations

i. $(m)=1,2,3,\ldots\ldots,m.$

ii. $(A_p)=A_1,A_2,A_3,\ldots\ldots A_p.$

iii. $[(A_p)]=A_1,A_2,A_3\ldots\ldots A_p.$

iv. $[(A_p)]_n=(A_1)_n\,(A_2)_n,(A_3)_n\ldots\ldots(A_p)_n.$

v. $\Delta(a,b)=\dfrac{b}{a},\dfrac{b+1}{a},\ldots\ldots\dfrac{b+a-1}{a}.$

vi. $\Gamma(a\pm b)=\Gamma(a+b)\Gamma(a-b).$

vii. $\Gamma_*\Gamma_*(a+b)=\Gamma(a+b)\Gamma(a+b).$

3. Recurrence Relations

Now, we shall derive some recurrence relations for the generalized hypergeometric polynomial set S_n (x, y).

(A) Form (1.2), we arrive at

$$S_n\left(x^{g_1},y^{g_2}\right)=\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times\frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}\lambda^m\,\lambda_1^{m_1}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_v)\right]_{m_2}\,m!\;\,m_1!}$$

$$\times \frac{\lambda_2^{m_2} \lambda_3^{n-m-e_1m_1-e_2m_2} \, y^{mg_2+e_1m_1g_2-e_2m_2g_2} \, x^{ne_3g_1-me_3g_1-e_1e_3g_1^{m_1}1-(e_3-1)e_2g_1m_2}}{m_2! \qquad (n-m-e_1m_1-e_2m_2)} \quad \ldots (3.1)$$

Differentiating with respect to x, we achieve

$$\frac{\partial}{\partial x} S_n\left(x^{g_1}, y^{g_2}\right) = (e_3 g_1) \sum_{m=0}^{n-1} \sum_{m_1=0}^{\left[\frac{n-1-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-1-m-e_1m_1}{e_2}\right]} \frac{\lambda^m \ \lambda_1^{m_1}}{m! \ \ m_1!}$$

$$\times \frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2} \left[(A_h)\right]_{n-m-e_1m_1-e_2m_2} \left[(G_r)\right]_{m_1}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2} \left[(B_k)\right]_{n-m-e_1m_1-e_2m_2} \left[(H_s)\right]_{m_1}}$$

$$\times \frac{\left[(C_u)\right]_{m_2} \lambda_2^{m_2} \lambda_3^{n-m-e_1m_1-e_2m_2-1} \, y^{mg_2+e_1m_1g_1-e_2m_2g_2} \, x^{xe_3g_1-me_3g_1-e_1e_3g_1m_1-(e_3-1)e_2m_2g_1}}{\left[(D_v)\right]_{m_2} (n-1-m-e_1m_1-e_2m_2)! \qquad m_2!}$$

$$+ e_2 g_1 m_2 \quad S_n\left(x^{g_1}, y^{g_2}\right) \quad \ldots (3.2)$$

Now $S^{\lambda;\lambda_1;\lambda_2;\lambda_3;(a_p)+1;(A_h)+1}_{n-1,m;m_1;m_2;(b_q)+1;(B_k)+1}\left(x^{g_1}, y^{g_2}\right)$

$$= \sum_{m=0}^{n-1} \sum_{m_1=0}^{\left[\frac{n-1-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-1-m-e_1m_1}{e_2}\right]} \frac{\left[(a_p)+1\right]_{n-1-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)+1\right]_{n-1-m-e_1m_1-(e_2-1)m_2}}$$

$$\times \frac{\left[(A_h)+1\right]_{n-1-m-e_1m_1-e_2m_2} \left[(G_r)\right]_{m_1} \left[(C_u)\right]_{m_2} \lambda^m \ \lambda_1^{m_1}}{\left[(B_k)+1\right]_{n-1-m-e_1m_1-e_2m_2} \left[(H_s)\right]_{m_1} \left[(D_v)\right]_{m_2} m! \ \ m_1!}$$

$$\therefore \ \times \frac{\lambda_2^{m_2} \lambda_3^{n-m-1-e_1m_1-e_2m_2} \, y^{mg_2-e_1m_1g_2} \left(x^{e_3}\right)^{g_1(n-1-m-e_1m_1-e_2m_1)-1} \, x^{e_2g_1m_1}}{m_2! \quad (n-1-m-e_1m_1-e_2m_2)! \quad \left(\lambda_3 y^{e_2g_1}\right)^{m_2}}$$

$$+ e_2 g_1 m_2 \quad S_n\left(x^{g_1}, y^{g_2}\right)$$

$$\frac{\left[(a_p)\right]\left[(A_h)\right] \lambda_3 x^{e_3g_1-1}}{\left[(b_q)\right]\left[(B_k)\right]} \ S^{(a_p)+1;(A_h)+1}_{n-1(b_q)+1;(B_k)+1}\left(g^{g_1}, y^{g_2}\right)$$

$$= \sum_{m=0}^{n-1} \sum_{m_1=0}^{\left[\frac{n-1-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-1-m-e_1m_1}{e_2}\right]} \frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times \frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2} \left[(G_r)\right]_{m_1} \left[(C_u)\right]_{m_2} \lambda^m \lambda_1^{m_1} \lambda_2^{m_2}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2} \left[(H_s)\right]_{m_1} \left[(D_v)\right]_{m_2} m! \ m_1! \ m_2!}$$

$$\times \frac{\left(\lambda_3 x^{e_3 x_1}\right)^{n-m-1-e_1m_1-e_2m_2} x^{e_2m_2} y^{mg_2+e_1m_1g_2-e_2m_2g_2}}{\left(n-1-m-e_1m_1-e_2m_2\right)!}$$

$$+e_2m_2g_2 \quad S_n\left(x^{g_1}, y^{g_2}\right) \qquad \ldots (3.3)$$

From (3.2) and (3.3), we achieve

$$\frac{\partial}{\partial x} S_n\left(x^{g_1}, y^{g_2}\right)$$

$$= \frac{\left[(a_p)\right]_n \left[(A_h)\right]_n \lambda_3 g_1 e_3 x^{e_3 g_1 - 1}}{\left[(b_q)\right]_n \left[(B_k)\right]_n} S_{n-1(b_q)+1;(B_k)+1}^{(a_p)+1;(A_h)+1}\left(g^{g_1}, y^{g_2}\right)$$

where $n \geq 1$ … (3.4)

Corollary: On putting $g_1 = 1 = g_2$ in (3.4), we get

$$\frac{\partial}{\partial x} S_n(x, y) = \frac{\left[(a_p)\right]\left[(A_h)\right]\lambda_3 e_3 g_1 x^{e_3 - 1}}{\left[(b_q)\right]\left[(B_k)\right]} S_{n-1(b_q)+1;(B_k)+1}^{(a_p)+1;(A_h)+1}(g, y)$$

where $n \geq 1$ … (3.5)

Particular Cases of (3.5):

(i) On Putting $p = 0 = q = h = k = u = v$; $m = 1 = m_1 = e_1 = e_3 = \lambda$; $\lambda_3 = 1 = e_2$;

$\lambda_2 = -1$, $y = x$, we get

$$\frac{d}{dx} H_n(x) = 2n \ \ H_{n-1}(x) = \frac{1}{n!} H_n(x)$$

Where $H_n(x)$ are the Hermite polynomials.

(ii) If we take $p = 0 = q = h = k = u$; $v = 1 = m = m_1 = e_1 = \lambda_2 = e_3$; $D_1 = 1$; $\lambda_3 = 1$; $y = 2x$, $e_2 = 2$, and $\frac{x}{\sqrt{x^2 - 1}}$ for x, we get

$$\left(x^2 - 1\right)\frac{d}{dx} P_n(x) = n\left[x \ P_n(x) - P_{n-1}(x)\right] = \left(x^2 - 1\right)^{\frac{-n}{2}} P_n(x)$$

Where $P_n(x)$ are the Legendre Polynomial.

(iii) On taking $h = 0 = u$; $k = 1 = v = e_3$; $D_1 = 1 + \beta$ in (3.5), we get

$$(x+1)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = n\,P_n^{(\alpha,\beta)}(x) + (n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x)$$

Where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

(iv) On putting $h = 0 = u$; $k = 1 = v = e_3$; $B_1 = 1+ \beta$; $D_1 = 1 + \alpha$ and instead of x and y, We get,

$$(x-1)P_n^{(\alpha,\beta)}(x) = n\,P_n^{(\alpha,\beta)}(x) - (n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x)$$

Where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

(v) On making the substitutions $h = 0 = u$; $k = 1 = v = e_3 = y$; and writing for x and y, we get

$$\frac{d}{dx}C_n^{\lambda}(x) = 2\lambda\,C_{n-1}^{\lambda+1}(x)$$

where $C_n^{\lambda}(x)$ are the Gagenbauer polynomials.

(B) Equation (3.4) can be written as

$$\left(x^{1-e_3g_1}\frac{\partial}{\partial x}\right)S_n\left(x^{g_1},y^{g_2}\right) = \lambda_3 e_3 g_1 \frac{\left[(a_p)\right]\left[(A_h)\right]}{\left[(b_q)\right]\left[(B_k)\right]}$$

$$\times\ldots\ldots\times\ S_{n-1,(b_q)+1,(B_k)+1}^{(a_p)+1,(A_h)+1}\left(g^{g_1},y^{g_2}\right) \qquad \ldots (3.6)$$

Differentiating successively m times, we get

$$\left(x^{1-e_3g_1}\frac{\partial}{\partial x}\right)^m S_n\left(x^{g_1},y^{g_2}\right) = \frac{(\lambda_3 e_3 g_1)^m\left[(a_p)\right]_m\left[(A_h)\right]_m}{\left[(b_q)\right]_m\left[(B_k)\right]_m}$$

$$\times\ldots\ldots\times\ S_{n-m,(b_q)+m,(B_k)+m}^{(a_p)+m,(A_h)+m}\left(g^{g_1},y^{g_2}\right)$$

where $n \geq m$ … (3.7)

Corollary: On putting $g_1 = 1 = g_2$ we achieve

$$\left(x^{1-e_3}\frac{\partial}{\partial x}\right)^m S_n(x,y) = \frac{(\lambda_3 e_3)^m\left[(a_p)\right]_m\left[(A_h)\right]_m}{\left[(b_q)\right]_m\left[(B_k)\right]_m}$$

$$\times\ldots\ldots\times\ S_{n-m,(b_q)+m,(B_k)+m}^{(a_p)+m,(A_h)+m}(g,y) \qquad \ldots (3.8)$$

where $n \geq m$ as m is non-negative integer.

Particular Cases of (3.8):

(i) On Putting $p = 0 = q = h = k = u = v$; $m = 1 = m_1 = e_1 = e_3 = \lambda$; $\lambda_3 = 1 = e_2$;

$\lambda_2 = -1$, $y = x$, we get

$$\frac{d^m}{dx^m} H_n(x) = \frac{2^n n!}{(n-m)!} H_{n-m}(x)$$

where $H_n(x)$ are the Hermite polynomials. [3]

(ii) If we take $p = 0 = q = h = k = u$; $v = 1 = m = m_1 = e_1 = \lambda_2 = e_3$; $D_1 = 1$; $\lambda_3 = 1$;

$y = 2x$, $e_2 = 2$, and $\frac{x}{\sqrt{x^2-1}}$ for x, we get

$$\frac{d^m}{dx^m}\left\{(x^2-1)^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{x^2-1}}\right)\right\} = \frac{n!(x^2-1)^{\frac{n-m}{2}}}{(n-m)!} P_{n-m}\left(\frac{x}{\sqrt{x^2-1}}\right)$$

where $P_n(x)$ are the Legendre polynomials.[3]

(iii) On taking $h = 0 = u$; $k = 1 = v = e_3$; , $D_1 = 1 + \beta$ in (3.8), we get

$$\frac{d^m}{dx^m}(1-x)^n P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right) = \frac{(1+\beta)_n (1-x)^{n-m}}{(1+\beta)_{n-m}} P_{n-m}^{(n+m,\beta)}\left(\frac{1+x}{1-x}\right)$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

(iv) On putting $h = 0 = u$; $k = 1 = v = e_3$; $B_1 = 1+\beta$; $D_1 = 1 + \alpha$ and instead of x and y, we get

$$\frac{d^m}{dx^m}\left\{(x-1)^m P_n^{(\alpha,\beta)}\left(\frac{x+1}{x-1}\right)\right\} = \frac{(1+\alpha)_n (x-1)^{n-m}}{(1+\alpha)_{n-m}} P_{n-m}^{(\alpha,\,\beta+m)}\left(\frac{x+1}{x-1}\right)$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials. [2]

(v) On making the substitutions $h = 0 = u$; $k = 1 = v = e_3 = y$; and writing for x and y, we get

$$\frac{d^m}{dx^m}\left\{(x-1)^m C_n^{\lambda}\left(\frac{x+1}{x-1}\right)\right\} = \frac{(2\lambda)_n x^{n-m}}{(n-m)}\ {}_2F_1\left[\begin{matrix} -n+m, \frac{1}{2}-\lambda-n; & \\ & \frac{1}{\alpha} \\ \lambda+\frac{1}{2}; & \end{matrix}\right]$$

where $C_n^{\lambda}(x)$ are the Gagenbauer polynomials. [2]

(C) Differentiating both sides of (3.1) partially with respect to *y*, we have

$$\frac{\partial}{\partial x} S_n\left(x^{g_1}, y^{g_2}\right) = \left(mg_2 + e_1 m_1 g_2 - e_2 m_2 g_2\right) \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \sum_{m_2=0}^{\left[\frac{n-m-e_1 m_1}{e_2}\right]}$$

$$\times \frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}}$$

$$\times \frac{\left[(C_u)\right]_{m_2}\lambda_1^{m_1}\lambda^m\lambda_3^{m_2}\lambda_3^{n-m-e_1m_1-e_2m_2}}{\left[(D_v)\right]_{m_2}\quad m!\quad m_1!\quad m_2!}$$

$$\times \frac{x^{ne_3g_1-me_3g_1-e_1e_3g_1m_1-(e_3-1)e_2g_1m_2}y^{mg_2+e_1m_1g_2-e_2m_2g_2-1}}{(n-m-e_1m_1-e_2m_2)!}$$

$$= mg_2\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times \frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}\lambda^m\ \lambda_1^{m_1}\lambda_2^{m_2}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_v)\right]_{m_2}\ m!\ \ m_2!}$$

$$\times \frac{\lambda_3^{n-m-e_1m_1-e_2m_2}x^{ne_3g_1-me_3g_1-e_1e_3g_1m_1-(e_3-1)e_2g_1m_2}y^{mg_2+e_1m_1g_2-e_2m_2g_{2-1}}}{(m-1)!\quad (n-m-e_1m_1-e_2m_2)!}$$

$$+e_1g_2\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times \frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}\lambda^m\ \lambda_1^{m_1}\lambda_2^{m_2}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_v)\right]_{m_2}\ m!\ \ (m_1-1)m_2!}$$

$$\times \frac{\lambda_3^{n-m-e_1m_1-e_2m_2}x^{ne_3g_1-me_3g_1-e_1e_3g_1m_1-(e_3-1)e_2g_1m_2}y^{mg_2+e_1m_1g_2-e_2m_2g_2-1}}{(n-m-e_1m_1-e_2m_2)!}$$

$$-e_2g_2\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\frac{\left[(a_p)\right]_{n-m-e_1m_1-(e_2-1)m_2}}{\left[(b_q)\right]_{n-m-e_1m_1-(e_2-1)m_2}}$$

$$\times \frac{\left[(A_h)\right]_{n-m-e_1m_1-e_2m_2}\left[(G_r)\right]_{m_1}\left[(C_u)\right]_{m_2}\lambda^m\ \lambda_1^{m_1}\lambda_2^{m_2}}{\left[(B_k)\right]_{n-m-e_1m_1-e_2m_2}\left[(H_s)\right]_{m_1}\left[(D_v)\right]_{m_2}\ m!\ \ m_1!\ \ (m_1-1)!}$$

$$\times \frac{\lambda_3^{n-m-e_1m_1-e_2m_2} x^{ne_3g_1-me_3g_1-e_1e_3g_1m_1+(e_3-1)e_2g_2m_2} y^{mg_2+e_1m_1g_2-e_2m_2g_2-1}}{(n-m-e_1m_1-e_2m_2)!}$$

$$= \frac{\left[(a_p)\right]_{g_2}}{\left[(b_q)\right] y^{e_2g_2+e_1g_1+m}} \left[m\lambda S_{n-m,(b_q)+1}^{(a_p)+1}\left(x^{g_1}, y^{g_2}\right) + e_1\lambda_1 \frac{\left[(G_r)\right]}{\left[(H_s)\right]} \right.$$

$$\left. \times S_{n-m_1,(b_q)+1,(H_s)+1}^{(a_p)+1,(G_r)+1}\left(x^{g_1}, y^{g_2}\right) - e_2\lambda_2 \frac{\left[(C_u)\right]}{\left[(D_v)\right]} S_{n-m_2,(b_q)+1,(D_v)+1}^{(a_p)+1,(C_u)+1}\left(x^{g_1}, y^{g_2}\right) \right]$$

Hence $\left(y^{e_2g_2+e_1g_2+m} \frac{\partial}{\partial y} \right) S_n\left(x^{g_1}, y^{g_2}\right)$

$$= \frac{m\lambda g_2\left[(a_p)\right]}{\left[(b_q)\right]} S_{n-m,(b_q)+1}^{(a_p)+1}\left(x^{g_1}, y^{g_2}\right) + \frac{e_1\lambda_1 g_1\left[(a_p)\right]\left[(G_r)\right]}{\left[(b_q)\right]\left[(H_s)\right]}$$

$$\times S_{n-m_1,(b_q)+1,(H_s)+1}^{(a_p)+1,(G_r)+1}\left(x^{g_1}, y^{g_2}\right) - \frac{e_2\lambda_2 g_2\left[(a_p)\right]\left[(C_u)\right]}{\left[(b_q)\right]\left[(D_v)\right]}$$

$$\times S_{n-m_2,(b_q)+1,(D_v)+1}^{(a_p)+1,(C_u)+1}\left(x^{g_1}, y^{g_2}\right) \qquad \dots (3.9)$$

where $n \geq m, m_1, m_2$

Corollary: If we take $g_1 = 1 = g_2$ we achieve

$$\left(y^{e_2+e_1+m} \frac{\partial}{\partial y} \right) S_n(x,y) = \frac{m\lambda\left[(a_p)\right]}{\left[(b_q)\right]} S_{n-m}(x,y)$$

$$+ \frac{e_1\lambda_1\left[(a_p)\right]\left[(G_r)\right]}{\left[(b_q)\right]\left[(H_s)\right]} S_{n-m,(b_q)+1,(H_s)+1}^{(a_p)+1,(G_r)+1}(x,y) - \frac{e_2\lambda_2\left[(a_p)\right]\left[(C_u)\right]}{\left[(b_q)\right]\left[(D_v)\right]}$$

$$\times S_{n-m_2,(b_q)+1,(D_v)+1}^{(a_p)+1,(C_u)+1}(x,y) \qquad \dots (3.10)$$

Particular Cases of (3.10):

(ii) On Putting $h = 0 = k = u$; $v = 1 = n = \lambda_3$; $\lambda_2 = -1$, for y, we achieve

$$\frac{d}{dy}L_n^{(\alpha)}(y) = -L_{n-1}^{\alpha+1}(y)$$

where $L_n^{(\alpha)}$ are the generalized Laguerre polynomials.

4. Conclusion
In this paper we have obtained many interesting new recurrence relations for the generalized hypergeometric polynomial set $S_n(x,y)$ followed by important and interesting particular cases. Out of these particular results some of them stand for well known and some of them are believed to be new. These are at most important for mathematicians, scientists and engineers.

5. References
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