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Purity relative to a cyclic module

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Abstract
We study relative projectivity and injectivity classes of exact sequences with respect to the classes of cyclic modules. A characterization of cyclic pure exact sequences has given in terms of exactness of a certain sequence of submodules of the modules appearing in the given sequence. We also study the concept of relative divisibility of elements in submodules (known as RD- purity). We give the generalization of the proposition of Stenstrom. We characterize the preservation of exactness by cyclic modules $R|I$ where I be a left ideal of the ring R. We also relate the $M-$ purity in Quasi- projective module [9]. We also try to define $\boldsymbol{\mathcal{A}}-$Copure for a class $\mathcal{M}$ of modules and co-relate $\mathcal{A}-$Copure injective or $\mathcal{A}-$Copure projective with it. We derive some results dualize certain results of R.B. Warfield.

Keywords: Left $R-$ modules, Ideals, Ideal purity, cyclic module, $\mathcal{A}-$Copure, $RD-$ purity, cocyclic copurity, $\mathcal{A}-$Copure injective or $\mathcal{A}-$Copure projective modules

Introduction

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. In similar manner module- purity plays an important role in the study of $R-$ module categories. The aim of the present paper is to study some aspects of purity relative to a fixed cyclic module $R|I$ for a left ideal I. We try to give the *characterization of cyclic pure exact sequences has given in terms of exactness of a certain sequence of submodules of the modules appearing in the given sequence. We also study the concept of relative divisibility of elements in submodules (known as* RD- purity*). We* give the generalization of the proposition of Stenstrom [13]. We characterize the preservation of exactness by cyclic modules $R|I$ where I be a left ideal of the ring R. We also relate the $M-$ purity in Quasi- projective module [9]. We also try to define $\mathcal{A}-$Copure for a class $\mathcal{M}$ of modules and corelate $\mathcal{A}-$Copure injective or $\mathcal{A}-$Copure projective with it. We derive some results dualize certain results of R.B. Warfield [15]. In this paper R refers to a ring with identity, which need not be necessarily commutative. Also by an $R-$ module we always mean a left $R-$ module.

Definition1.1. An exact sequence is said to be M- pure

$$\begin{array}{c} M \\ \downarrow \\ 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \end{array}$$

if given $f: M \longrightarrow C$, there exists $f': M \longrightarrow B$ such that $(\beta o f') = f$.
Now we shall consider purity with respect to a fixed cyclic left $R-$module $R|I$. Where R is a left $R-$ module and I is a left ideal of R [6].

Proposition1.2: For a left ideal I, a submodule K of M is $R|I-$ pure if and only if given $m \in M$ such that $Im \subseteq K$. there exists $m' \in M$ such that $Im' = 0$ and $(m - m') \in K$, where $R|I$ is a fixed cyclc left module.

Proof: Given the lower sequence, $f: R|I \longrightarrow M|K$ and the given above condition, we construct the following diagram by projectivity of R.

$$0 \to I \overset{\alpha}{\to} R \overset{\beta}{\to} R|I \to 0$$
$$\downarrow \downarrow \swarrow \downarrow$$
$$0 \to K \overset{\alpha'}{\to} M \overset{\beta'}{\to} M|K \to 0$$

Let $f(\bar{1}) = m + K$ $(\bar{1} \in R|I)$. Since $f(\bar{1}) = f(1 + I)$,
$I(m + K) = If(\bar{1}) = I\big(f(1 + I)\big) = \{rf(1 + I)|\, r \in I\}$
$= \{f(r + I)|\, r \in I\} = \{f(0)\} = \{0\}$, Therefore, $Im \subseteq K \Rightarrow \exists\, m'$ with $Im' = 0$ and $(m - m') \in K$.
We define $h: R|I \longrightarrow M$ by $h(r) = rm'$. We want to show it is well defined.
If $\bar{r} = \bar{s} \Rightarrow (r - s) \in I \Rightarrow (r - s)m' = 0$
$\Rightarrow rm' = sm' \Rightarrow h(\bar{r}) = h(\bar{s})$. Now,
$(\beta' oh)(\bar{r}) = \beta'(rm') = r\beta'(m') = r(m' + K) = r(m + K)$
$= rf(\bar{1}) = f\big(r(\bar{1})\big) = f(\bar{r})\ \forall\, \bar{r} \in R|I$, therefore sequence is $R|I -$ pure since, $(\beta oh) = f$.

Conversely, given $m \in M$ with $Im \subseteq K$, define $f: R|I \longrightarrow M|K$ by $f(r) = rm + K$, then f is well defined as $r \in I \Rightarrow rm \in K$. By $R|I -$ purity of the lower sequence,

$h: R|I \longrightarrow M$ by $h(r) = rm'$ exists with $(\beta' oh) = f$.
Take $m' = h(\bar{1})$, then $Im' = Ih(\bar{1}) = \{h(\bar{\imath})\,|i \in I\} = \{h(0)\} = 0$.
Also, $(m' + K) = h(\bar{1}) + K = \beta' oh(\bar{1}) = f(\bar{1}) = m + K$ that is $(m - m') \in K$.

Definition1.3: Let M be a left $R -$ module. For a two sided ideal I.
we define $M[I] = \{m \in M\ s.t.\ Im = 0\}$. Then $M[I]$ is an $R -$ submodule of M.
The following theorem is analogue of the corresponding results on purity in abelian groups [12].

Theorem1.4: Suppose that I be a fixed two sided ideal of R and K is $R|I -$ pure in M, where $R|I$ is a fixed cyclic left module.
Then the following conditions are equivalent:
(a). $0 \longrightarrow K \overset{\alpha}{\to} M \overset{\beta}{\to} M|K \longrightarrow 0$ is an $R|I -$ pure exact sequence.
(b). $0 \longrightarrow K[I] \overset{\alpha'}{\to} M[I] \overset{\beta'}{\to} (M|K)[I] \longrightarrow 0$ is exact.
(c). $0 \longrightarrow K\Big|K[I] \overset{\alpha*}{\to} M\Big|\, M[I] \overset{\beta*}{\to} (M|K)|(M|K)[I] \longrightarrow 0$ is exact,
Where $M[I] = \{m \in M\ |Im = 0\}$ And α^*, β^* are maps induced by α, β respectively.

Proof: $(a) \Longrightarrow (b)$. To show (b) is exact, that is to show that $Image(\alpha') = Ker(\beta')$, let $m \epsilon\ Image(\alpha')$, therefore, $m = \alpha'(k)$ for some $k \in K[I]$,
$I\big(\alpha(k)\big) = \alpha(Ik) = \alpha.0 = 0$ (since $Ik = 0$).
Therefore, $\alpha(k) \in K[I]$, hence, $Image(\alpha') \subseteq Image(\alpha) \cap M[I]$
$= \ker(\beta) \cap M[I] = ker(\beta')$, …… (1)
$ker(\beta') = \{m \in M[I] \mid \beta'(\text{m}) = 0\} = \{m \in M\ |\beta(m) = 0, Im = 0\}$
$= \ker(\beta) \cap M[I] = Image(\alpha) \cap M[I]$, (Since $\ker(\beta) = Image(\alpha)$).
Take $\alpha(k) \in Image(\alpha) \cap M[I]$, therefore,
$I\alpha(k) = 0 \Longrightarrow \alpha(Ik) = 0$, Therefore, $Ik = 0$ (Since α is injective).
Therefore, $k \in K[I]$ and $\alpha(k) \in Image(\alpha')$
$\Longrightarrow Image(\alpha) \cap M[I] = ker(\beta') \subseteq Image(\alpha')$…………(2)
From these two equations (1) and (2) we get $ker(\beta') = Image(\alpha')$.
Now to show that β^* is epic, that is to show that for each element $(m + K) \in (M|K)[I]$
(That is $I(m + K) = 0$), there exists an element $m' \in M[I]$
Such that $\beta'(m') = m' + K = m + K$.
Since $I(m + K) = 0 \Longrightarrow Im + K = 0 \Longrightarrow Im \subseteq K$, therefore, from the proposition [**1.2**] there exists $m' \in M$ such that $Im' = 0$ and $m' \in M[I]$. (Since $Im' = 0$), therefore, β' is epic.
$(b) \Longrightarrow (a)$. Suppose (b) is exact then we have to show that (a) is $R|I -$ pure, that is for given $m \in M$ with $Im \subseteq K$ there exists an element $m' \in M$ such that $Im' = 0$ and $(m - m') \in K$. Since $Im \subseteq K \Longrightarrow (m + K) = 0$, therefore, $(m + K) \in (M|K)[I]$, and as β' is epic.

Therefore, $(m + K) = \beta'(m')$ for some $m' \in M[I]$, therefore, $Im' = 0$, and $(m - m') \in K$. Hence, $0 \longrightarrow K \overset{\alpha}{\to} M \overset{\beta}{\to} M|K \longrightarrow 0$ is an $R|I -$ pure exact sequence.
$(b) \Leftrightarrow (\boldsymbol{c})$ is proved by using of 3 X 3 lemma [6].

$$0\ 0\ 0$$
$$\downarrow \downarrow \downarrow$$
$$0 \longrightarrow K[I] \overset{\alpha'}{\to} M[I] \overset{\beta'}{\to} (M|K)[I] \longrightarrow 0$$
$$\downarrow \downarrow \downarrow$$

$$0 \longrightarrow K \overset{\alpha}{\rightarrow} M \overset{\beta}{\rightarrow} M|K \longrightarrow 0$$
$$\downarrow\downarrow\downarrow$$
$$0 \longrightarrow K \Big| K[I] \overset{\alpha*}{\rightarrow} M \Big| M[I] \overset{\beta*}{\rightarrow} (M|K)|(M|K)[I] \longrightarrow 0$$
$$\downarrow\downarrow\downarrow$$
$$0\ 0\ 0$$

We have called an exact sequence $\mathcal{A}$ −Copure for a class $\mathcal{M}$ of modules if objects of $\mathcal{M}$ are injective with respect to the exact sequence $0 \longrightarrow K \overset{\alpha}{\rightarrow} M \overset{\beta}{\rightarrow} N \longrightarrow 0$, Where $K, M, N \in \mathcal{M}$, and we call a module $\mathcal{A}$ −Copure injective or $\mathcal{A}$ −Copure projective if it is respectively injective or projective relative to $\mathcal{A}$ −Copure sequences. The following results dualize certain results of R.B. Warfield [15], and at the same time, they generalize certain results in purity in abelian group in Fuchs [12] to module categories.

Definition1.5: An $R-$ module M is said to be Concyclic, if it is a submodule of $E(S)$, for some simple module S. These are nothing but the subdirectly irreducible modules**.**

Proposition1.6: Let S be a class of left R- modules containing the modules $E(S_i)$ where S_i are a representative class of of all simple modules, such that there is a subclass S^* which is a set with the property that for any $M \in S$, there is an $N \in S^*$ with $N \approx M$, then for any module A, there is an $S-$ copure sequence $0 \rightarrow A \rightarrow C \rightarrow C' \rightarrow 0$ such that C is a direct product of copies of modles in S.

Proof: Let Λ be the set of pairs (M, f) with $M \in S^*$ and $f \in Hom(A, M)$ and for each $\lambda \in \Lambda$ denote the corresponding M and f by M_λ and f_λ. Let $C = \prod_{\lambda \in \Lambda} M_\lambda$ and let $f: A \rightarrow C$ be the product map of the maps f_λ. Since $p_\lambda o\emptyset = f_\lambda$, then $\emptyset$ is injective because A can be embedded into a direct product of $E(S_i)'s$.

Theorem1.7: A module P is S-copure injective if and only if it is a direct summand of direct product of copies of modules in S.
Notes: Proof of this theorem is similar as in [8].

Corollary1.8: Let S be the class of all cocyclic modules, then for any module A there is an $S-$ copure sequence $0 \rightarrow A \rightarrow N \rightarrow N' \rightarrow 0$ such that N is the direct product of copies of cocyclic modules in S.
Notes: Proof of this theorem is similar as in [8].

Corollary1.9: A left $R-$ module M is cocyclic copure injective if and only if it is a direct summand of a direct product of cocyclic modules.
Notes: Proof of this theorem is similar as in [8].

Proposition1.10: A submodule K of an R- module M is RD-pure in M if and only if given $m \in M$ and $\lambda \in (K: m)$, there exists $m' \in M$ such that $\lambda m' = 0$ and $(m - m') \in K$.
Now we take I vary over all left ideals and give the generalization of the following proposition of Stenstrom [13].

Proposition1.11: K is $\vartheta-$ pure in M, where ϑ is the class of all cyclic, if and only if given $\overline{m} \in M|K$, there exists $m' \in M$ such that $(m - m') \in K$ and $Ann(m') = Ann(\overline{m})$.

Proposition1.12: A submodule $K|J$ of the cyclic module $R|J$, is $R|I-$ pure in $R|J$ if and only if given $r \in R$ such that $Ir \subseteq K$, there exists $r' \in R$ such that $Ir' \subseteq J$ and $(r - r') \in K$.

Proposition1.13: A cyclic module $R|J$, is $R|I-$ regular that is all of its submodules are $R|I-$ pure in it, if and only if $r \in R$, there exists $r' \in R$ such that $Ir' \subseteq J$ and $(r - r') \in J + Ir$.

Proof: Given the condition, if $K|J \subseteq R|J$, then for $r \in R$ such that $Ir \subseteq K$, there exists $r' \in R$ such that $Ir' \subseteq J$ and $(r - r') \in J + Ir \subseteq K$.
Conversely, given $r \in R$, $Ir \subseteq J + Ir$ and as $(J + Ir)|J$ is $R|I-$ pure in $R|J$, there exists $r' \in R$ such that $Ir' \subseteq J$ and $(r - r') \in J + Ir \subseteq K$.
Note: Observe that a module M is Quasi- projective if and only if each submodule N of M is $M-$ pure in M [9].

Proposition1.14: A cyclic module $R|I$ is Quasi- projective if and only if given $r \in R$, there exists $i_r \in I$ such that $I(r - i_r) \subseteq I$.

Proof: By the previous proposition $R|I$ is Quasi- projective if and only if given $r \in R$, there exists $r' \in R$ such that $Ir' \subseteq I$. and $(r - r') \in I + Ir$. If $R|I$ is Quasi- projective and $r \in R$, then choose $r' \in R$ such that $Ir' \subseteq I$. and $(r - r') \in I + Ir$.
Now if $(r - r') = i_1 + i_2 r$, then $I(r - i_2 r) = I(r' + i_1) \subseteq I$ and hence the condition holds.
Conversely, given the condition holds and $r \in R$,we take $r' = (r - i_r r)$, then $(r - r') = i_r \subseteq I + Ir$ and $Ir' = I(r - i_r r) \subseteq I$ and so, $R|I$ is Quasi- projective.

Proposition1.15: If I is a two sided ideal, then $R|I$ is a Quasi-projective module.

Remark1.16: Recall that a ring is called q^*- ring if all cyclic modules are quasi- projective (Koehler [16]). We get that a ring R *is a* q^*- ring if and only if given any $r \in R$ and any left ideal I, there exists $i_r \in I$ such that $I(r - i_r r) \subseteq I$.

Remark1.17: All commutative and duo rings are q^*- rings.

Proposition1.18: If I is a two sided ideal of a left q^*- ring R, then $R|I$ is a left q^*- ring.
Proof: Given $r \in R$ and a left ideal $J \subseteq I$, there exists $i_r \in J$ such that $J(r - i_r r) \subseteq J$. Then $J|I[(r + I) - (i_r + I)(r + I)] \subseteq J|I$ and hence the result follows.

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