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Further remarks on unitary equivalence of some classes of operators in Hilbert spaces

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Abstract
In this paper we investigate results on unitary equivalence of operators that include n-binormal, skew binormal and n-power-hyponormal operators acting on complex Hilbert space H.
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Keywords: n-binormal, isometry, skew binormal, n-power-hyponormal and unitary equivalence.

Introduction

In this paper, B (H) denotes the Banach algebra of all bounded linear operators acting on a complex Hilbert space H. Two operators acting A and B are said to be unitarily equivalent if there exists a unitary operator U such that $B=U^* AU$. Properties of similarity and unitary equivalence has been studied widely by a few authors among them B.M Nzimbi, G.P Porkariyal and S.K Moindi who studied unitary equivalence of normal operators; R.G Douglas who showed that quasi-similar normal operators are unitarily equivalent. Recently S.W Luketero and J.M Khalagai studied unitary equivalence of binormal and hyponormal operators whereas Wanjala Victor, R.K Obogi and M.O Okoya studied metric equivalence of n-normal operators. Messaoud Guesba and Mostefa Nadir considered unitary equivalence covering n-power-hyponormal operators while Krutani Rasimi, YIIdrita Seferi, Alit Ibraimi and Flamure Sadik reviewed unitary equivalence of skew n-binormal.

In this paper we continue with study of unitary equivalence of operators and endeavor to improve on the following results as advanced by the respective authors.

Theorem A [8, Theorem 3.1]
If S is an n-normal operator and $T \in B(H)$ is unitarily equivalent to S, then T is an n-normal operator

Theorem B [3, Proposition 1]
If $S, T \in B(H)$ are unitarily equivalent and T is n-power-hyponormal operator then so is S.

Theorem C [2, Theorem 2.4 part ii]
If S is unitary equivalent to T then S is skew n-binormal operator.

Notations, definitions and terminologies

An operator $T \in B(H)$ is said to be n-normal if $T^nT^* = T^*T^n$; normal if $T\,T^* = T^*T$. Note that every normal operator is n-normal for any positive integer n. An operator $T \in B(H)$ is said to be an isometry if $T^*T = I$ or equivalently if $\|Tx\| = \|x\|\ \forall x \in H$. $T \in B(H)$ is said to be a co-isometry if $T\,T^* = I$ and quasinormal if $[T^*, T^*T] = 0$. An operator $T \in B(H)$ is called partial isometry if $T = TT^*T$ and unitary if $T\,T^* = T^*T = I$.

An operator $T \in B(H)$ is called hyponormal if $T\ T^* \leq T^*T$ and co-hyponoormal if $T\ T^* \geq T^*T$. An operator $T \in B(H)$ is called n-power-hyponormal if $T^nT^* \leq T^*T^n$ and co-n-power-hyponormal if $T^nT^* \geq T^*T^n$ .An operator $T \in B(H)$ is called binormal if $[T^*T, TT^*] = 0$. The binormal operator T with $(T^*T)(TT^*) = (TT^*)(T^*T)$ is skew binormal operator if $(T^*TTT^*)T = T(TT^*T^*T)$ and is said to be n-binormal operator if it satisfy $T^*T^nT^nT^* = T^nT^*T^*T^n$. For $S, T \in B(H)$ the commutator of S and T is given by $[S,T] = ST - TS$. An operator A is said to be similar to another operator B if there exists an invertible operator S such that $A = S^{-1}BS$. Two operators S and T are said to be unitarily equivalent if there exists a unitary operator U such that $S = U^*TU$. S and T are unitarily equivalent if $U^*S^n = T^nU^*$.

It is worth noting the following relationships among a few classes of operators which will enable generalization of results.

$normal \subseteq quasinormal \subseteq binormal \subseteq n-binormal$

$normal \subseteq subnormal \subseteq hyponormal \subseteq n-power-hyponormal$

$normal \subseteq n-normal \subseteq hyponormal \subseteq n-power-hyponormal$

$unitary \subseteq isometry \subseteq partial\ isometry$

$unitary \subseteq co-isometry \subseteq partial\ isometry$

Main Results

Theorem 3.1

Let S be an n-binormal operator and $T \in B(H)$ be unitarily equivalent to S, then T is also n-binormal.

Proof

First note that $T = U^*SU$ and hence $T^* = U^*S^*U$.

$$
\begin{aligned}
T^*T^n &= U^*S^*UU^*S^nU \\
&= U^*S^*S^nU \\
&= U^*S^nS^*U \\
&= T^nU^*S^*U = T^nU^*US^* = T^nS^*
\end{aligned}
$$

Hence T is n-binormal.

Theorem 3.2

Let $A \in B(H)$ be an n-binormal operator and B be any operator such that either

- $A = UBU^*$ where U is an isometry.
- $A = U^*BU$ where U is co-isometry, then B is n-binormal.

Proof

- Since A is n-biormal it follows that $A^*A^nA^nA^* = A^nA^*A^*A^n$.....................1. Note also that $A^* = UB^*U^*$ and $A^n = UB^nU^*$. Left hand side of equation 1 gives $UB^*U^*UB^nU^*UB^nU^*UB^*U^* = UB^*B^nB^nB^*U^*$.....................2 while right hand side of equation 1 gives $UB^nU^*UB^*U^*UB^*U^*UB^nU^* = UB^nB^*B^*B^nU^*$..............3.Equating the two equations 1 and 2 we obtain $UB^*B^nB^nB^*U^* = UB^nB^*B^*B^nU^*$.............4. Pre-multiplying equation 4 by U^* and post-multiplying it with U we obtain $B^*B^nB^nB^* = B^nB^*B^*B^n$. Hence the conclusion that B is n-binormal.

- A being n-binormal implies $A^*A^nA^nA^* = A^nA^*A^*A^n$. The proof is similar to that part i where $A^* = U^*B^*U$ and $A^n = U^*B^nU$.

  $U^*B^*UU^*B^nUU^*B^nUU^*B^*U = U^*B^nUU^*B^*UU^*B^*UU^*B^nU$ Which simplifies to

  $U^*B^*B^nB^nB^*U = U^*B^nB^*B^*B^nU$. Now pre-multiplying this equation by U and post-multiplying it with U^* and bearing in mind that U is a co-isometry we obtain $B^*B^nB^nB^* = B^nB^*B^*B^n$. Hence B is n-binormal.

Remark 3.3

Theorems 3.2 and 3.4 advanced by [7] now become consequences of theorem 3.2 and can be summarized by the following corollary.

Corollary3.4

Let $A, B \in B(H)$ be such that A is binormal and either

- $A = UBU^*$ where U is an isometry or
- $A = U^*BU$ where U is a Co-isometry, then B is also binormal

Proof

It suffices to note that every class of binormal operators is n-binormal

Theorem 3.5

Let A be n-power-hyponormal and B be any operator such that either

- $A = UBU^*$ where U is an isometry or
- $A = U^*BU$ where U is a co-isometry, then B is n-power-hyponormal.

Proof

- Note that $A = UBU^*$ implies $A^* = UB^*U^*$ and $A^n = UB^nU^*$. Note that $A^nA^* = UB^nU^*UB^*U^* = UB^nB^*U^*$ and $A^*A^n = UB^*U^*UB^nU^* = UB^*B^nU^*$. Since A is n-hyponormal we have that $A^*A^n \geq A^nA^*$. That is $UB^*B^nU^* \geq UB^nB^*U^*$............1. Pre-multiplying by U^* and post-multiplying inequality 1 by U we obtain $U^*UB^*B^nU^*U \geq U^*UB^nB^*U^*U$ which simplify to $B^*B^n \geq B^nB^*$. Hence B is n-power-hyponormal.
- First note that $A = U^*BU$ implies $A^* = U^*B^*U$ and $A^n = U^*B^nU$. Now $A^nA^* = U^*B^nUU^*B^*U = U^*B^nB^*U$ and $A^*A^n = U^*B^*UU^*B^nU = U^*B^*B^nU$. Since A is n-power-hyponormal we have that $A^*A^n \geq A^nA^*$ which implies that $U^*B^*B^nU \geq U^*B^nB^*U$............2. Pre-multiplying by U and post-multiplying inequality 2 by U^* we obtain $UU^*B^*B^nUU^* \geq UU^*B^nB^*UU^*$ which simplify to $B^*B^n \geq B^nB^*$. Hence B is n-power-hyponormal

Remark3.6

S.W Luketero and J.M Khalagai proved the following result which becomes a consequence of Theorem 3.5 as stated above.

Theorem D

Let A be a hyponormal operator and B be another operator such that either

- $A = UBU^*$ where U is an isometry or
- $A = U^*BU$ where U is a co-isometry.

Then B is also hyponormal

Proof

It is enough to note that every hyponormal operator is n-hyponormal.

Theorem 3.7

Let $A, B \in B(H)$ be such that A is skew binormal and either

- $A = UBU^*$ where U is an isometry or
- $A = U^*BU$ where U is a co-isometry, then B is skew-binormal.

Proof

Recall that if A is skew binormal then $(A^*AAA^*)A = A(AA^*A^*A)$.

-

$$
\begin{aligned}
(A^*AAA^*)A &= (UB^*U^*UB\,U^*UB\,U^*UB^*U^*)UBU^* \\
&= UB^*BBB^*U^*UBU^* \\
&= UB^*BBB^*BU^*
\end{aligned}
$$

$$
\begin{aligned}
A(AA^*A^*A) &= UBU^*(UBU^*UB^*U^*UB^*U^*UBU^*) \\
&= UB\,U^*UBB^*B^*BU^* \\
&= UBBB^*B^*BU^*
\end{aligned}
$$

Equating the two gives $UB^*BBB^*BU^* = UBBB^*B^*BU^*$. Pre-multiplying by U^* and post-multiplying by U and noting that U is isometry gives $(B^*BBB^*)B = B(BB^*B^*B)$ which implies that B is skew-binormal.

- We make use of the operator equation $A = U^*BU$ where $A^* = U^*B^*U$.

$$
\begin{aligned}
(A^*AAA^*)A &= (U^*B^*UUBUUBUU^*B^*U)U^*BU \\
&= U^*B^*BBB^*UU^*BU \\
&= U^*B^*BBB^*BU
\end{aligned}
$$

Where U is a co-isometry. Similarly,

$$
\begin{aligned}
A(AA^*A^*A) &= U^*BU(U^*BUU^*B^*UU^*B^*UU^*BU) \\
&= U^*BUU^*BB^*B^*BU \\
&= U^*BBB^*B^*BU
\end{aligned}
$$

Equating the two gives $U^*B^*BBB^*BU = U^*BBB^*B^*BU$. Pre-multiplying by U and post-multiplying by U^* and noting that U is a co-isometry gives $(B^*BBB^*)B = B(BB^*B^*B)$ which implies that B is skew-binormal.

Remark 3.8

We note that since the class of unitary operators is contained in isometry and co-isometry, the following corollary which generalizes Theorem A, proposition B and Theorem C is immediate:

Corollary3.9

Let $A, B \in B(H)$ be such that either $A = UBU^*$ or $A = U^*BU$ where U is unitary and A belongs to any of the following classes of operators then so is B

- n-normal
- n-binormal
- Skew binormal
- n-power hyponormal

Corollary3.10

We note from all the results above that the following classes of operators are not only unitarily invariant but also isometrically and co-isometrically invariant.

- n-normal
- n-binormal
- Skew binormal
- n-power hyponormal

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