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Praveen Sharma
P.G. Student, Department of mathematics, University of Delhi, Delhi, India

Corresponding Author:
Praveen Sharma
P.G. Student, Department of mathematics, University of Delhi, Delhi, India

Schwarz lemma applications

Praveen Sharma

Abstract
In this Paper, we have studied the Schwarz's lemma and arbitrary analytic functions on the open unit disk D and we have proved that range of all the analytic functions on the open unit disk always lies in the closed annulus with centre origin and inner and outer radius depends on the modulus of the function at origin and also discuss its one more application.

Keywords: Connected, analytic, annulus, disk

1. Introduction
1.1 Analytic Function
Let G be an open set in the complex plane C then a function f defined on G is said to be analytic at a ∈ G if f is differentiable at z = a and f0(a) is continuous.
Note that: a function f defined on an open subset G of C is said to be analytic on G if f is analytic at each point of G.

1.2 Region
An subset G of C is said connected if any two points in G are joined by a continuous function whose image lies completely in G. An open connected subset G of C is said to be a region in C. [1]

Notation

1.3 Maximum Modulus Theorem
Let f be analytic function on some region G ⊆ C if there exists some a ∈ G such that

$$|f(z)| \leq |f(a)| \ \forall \ z \in G.$$

Then f must be a constant function.

2. Schwarz's lemma
Let f be analytic function from D into D (i.e $|f(z)| \leq 1 \ \forall z \in D$) mapping 0 to 0 then

$$\lim_{z \to 0} |\frac{f(z)}{z}| \leq 1 \ \ and \ \ |f(z)| \leq 1 \ \ \forall \ \ z \in \mathbb{D}$$

Moreover [2], equality occurs in one of the inequality above for some 0 6= z ∈ D then ∃ a constant c ∈ C with |c| = 1 with

$$F(w) = cw \quad \forall \ w \in D$$

Theorem 2.1. Let f: D → D be analytic function then

$$|f(z)| \in \overline{\text{ann}\{0, r(z), R(z)\}} \quad \forall z \in D$$

where r and R are function on the open unit disk D given by

$$r(z) = \frac{|f(0)| - |z|}{1 + |f(0)||z|} \ \ and \ \ R(z) = \frac{|f(0)| + |z|}{1 - |f(0)||z|} \quad \forall z \in D$$

Proof. • **Case1:** when $|f(0)| = 1$

Since

$$f(z) \in D \Rightarrow |f(z)| \leq 1 \ \forall z \in D$$

∴ f attains its maximum modulus in the open connected set D then by Maximum modulus theorem (1.3) f must be a constant function with modulus 1.

$|f(z)| = 1 \ \forall z \in D$

Then

$$r(z) = \frac{1 - |z|}{1 + |z|} \leq 1 \leq \frac{1 + |z|}{1 - |z|} = R(z) \ \ \forall z \in \mathbb{D}$$

$$\Rightarrow |f(z)| \in \overline{ann\left\{0, \frac{1 - |z|}{1 + |z|}, \frac{1 + |z|}{1 - |z|}\right\}} \qquad \forall \ z \in \mathbb{D}$$

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Case 2: when $|f(0)| = 0$

Since f is analytic function ⇒ $f(0) = 0$
Then f is analytic function mapping D into D mapping 0 to 0 Then By Schwarz's Lemma we have

$$|f(z)| \leq |z| \ \forall z \in D$$

$$\Rightarrow -|z| \leq |f(z)| \leq |z| \ \forall z \in D$$

Where

$$r(z) = -|z| \text{ and } R(z) = |z|$$

$$\Rightarrow |f(z)| \in \text{ann}\{0, -|z|, |z|\} \quad \forall z \in D$$

Case3: When |f(0)| 6= 0,1
Define a function h on D given by

$$h(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}$$

Clearly

$$h(0) = 0$$

To show $|h(z)| \leq 1 \ z \in D$

Since

$$0 < |f(0)| < 1 \Rightarrow |\overline{f(0)}| < \frac{1}{f(0)}$$

$$|h(z)| = \Big|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\Big| \leq \Big|\frac{f(z) - f(0)}{f(0) - f(z)}\Big| \leq 1 \ \ z \in \mathbb{D}$$

Now

∴ h is analytic function from the open unit disk D into D mapping 0 to 0. Then By Schwarz's Lemma we have

$$|h(z)| \leq |z| \ \ \forall z \in \mathbb{D}$$

$$\Rightarrow \Big|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\Big| \leq |z| \ \ \forall z \in \mathbb{D}$$

$$\Rightarrow |f(z) - f(0)\Big| \leq |z|(|1 - |\overline{f(0)}||f(z)|) \ \ \forall z \in \mathbb{D}$$

By Triangle Inequality we have

$||f(z)| - |f(0)|| \leq |f(z) - f(0)| \leq |z|(|1 - |f(0)||f(z)|) \ \forall z \in D$

$$\Rightarrow ||f(z)| - |f(0)|| \leq |z|(|1 - |\overline{f(0)}||f(z)| \leq |z|(1 + |\overline{f(0)}||f(z)|) \ \ \forall z \in \mathbb{D}$$

$$\Rightarrow -|z|(1 + |\overline{f(0)}||f(z)|) \leq |f(z)| - |f(0)| \leq |z|(1 + |\overline{f(0)}||f(z)|) \ \ \forall z \in \mathbb{D} \qquad (1)$$

Taking first two parts of inequality (1) them for each z ∈ D we have

$$-|z|(1 + |\overline{f(0)}||f(z)|) \leq |f(z)| - |f(0)|$$

$$\Rightarrow -|z| - |z||\overline{f(0)}||f(z)| \leq |f(z)| - |f(0)|$$

$$\Rightarrow |f(0)| - |z| \leq |f(z)|[1 + |z||\overline{f(0)}|]$$

$$\Rightarrow r(z) = \frac{|f(0)| - |z|}{1 + \overline{f(0)}|z|} \leq |f(z)|$$

$$\therefore r(z) \leq |f(z)| \ \ \forall z \in \mathbb{D} \qquad (2)$$

Taking last two inequalities of (1) for each z ∈ D we have

$$|f(z)| - |f(0)| \leq |z|(1 + |\overline{f(0)}||f(z)|)$$

$$\Rightarrow |f(z)| - |f(0)| \leq |z| + |\overline{f(0)}||f(z)|$$

$$\Rightarrow |f(z)|[1 - |\overline{f(0)}||f(z)|] \leq |f(0)| + |z|$$

$$\Rightarrow |f(z)| \leq \frac{|f(0)| + |z|}{1 - |\overline{f(0)}||f(z)|} = R(z)$$

$$\therefore |f(z)| \leq R(z) \ \ \forall z \in \mathbb{D} \qquad (3)$$

∴ from (2) and (3) we have

|f(z)| ∈ ann{0,r(z),R(z)}   ∀ z ∈ D

Theorem 2.2. Let f is a non constant analytic function on the disk whose real part is always non zero Then Show that

Ref (z) > 0 ∀z ∈ D

Moreover if f (0) = 1 Then

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|} \quad z \in \mathbb{D}$$

Proof. Suppose if possible Ref (z0) = 0 for some z0 ∈ D Define a function h on D by

h(z) = exp(−f(z)) ∀z ∈ D

Then
|h(z)| = |exp(−f(z)| = |exp(−Ref(z))| ≤ 1 ∀z ∈ D
(∵ Ref(z) ≥ 0)

Also

|h(z0)| = |exp(−f(z0))| = 1

∴ h attains its maximum modulus on the region D
By maximum modulus Theorem (1.3) h must be a constant function on the disk D ⇒ f must be a constant function on the disk D
Which is a contradiction since f is a non-constant analytic function

∴ Ref (z) > 0 ∀z ∈ D

Let f(0) = 1 Then Define a function g on D by

$$g(z) = \frac{f(z)-1}{f(z)+1} \;\; \forall z \in \mathbb{D}$$

$$g(0) = \frac{f(0)-1}{f(0)+1} = 0$$

Clearly

To show

|g(z)| ≤ 1 z ∈ D

Let z = x + ιy ∈ D where x > 0 (∵ Ref(z) > 0)
Then

$$|g(x+\iota y)| \leq \left|\frac{x-1+\iota y}{x+1+\iota y}\right|$$

$$\Rightarrow |g(x+\iota y)| = \left|\frac{x^2+y^2-1+2\iota y}{(x+1)^2+y^2}\right|$$

$$\Rightarrow |g(x+\iota y)|^2 \leq \frac{(x^2+y^2-1)^2+4y^2}{(x^2+y^2+1+2x)^2}$$

$$\Rightarrow |g(x+\iota y)|^2 \leq \frac{(x^2+y^2+1)^2-4x^2}{(x^2+y^2+1)^2+4x^2+4x(x^2+y^2+1)} \leq \frac{(x^2+y^2+1)^2-4x^2}{(x^2+y^2+1)^2+4x^2} \leq 1 \quad (\because x>0)$$

$$\therefore |g(z)| \leq 1 \;\; \forall z \in \mathbb{D}$$

∴ g is analytic function from D into D mapping 0 to 0

Then By Schwarz's Lemma we have

$$|g(z)| \leq |z| \ \forall z \in \mathbb{D}$$

$$\Rightarrow \left|\frac{f(z)-1}{f(z)+1}\right| \leq |z| \ \forall z \in \mathbb{D}$$

Then by Triangle inequality we have

$||f(z)| - 1| \leq |f(z) - 1| \leq |z|(|f(z) + 1|) \leq |z|(|f(z)| + 1) \ \forall z \in D$

$\Rightarrow ||f(z)| - 1| \leq |z|(|f(z)| + 1) \ \forall z \in D$

$\Rightarrow -|z|(1 + |f(z)|) \leq |f(z)| - 1 \leq |z|(1 + |f(z)|) \ \forall z \in D$ (1) By

Taking first two inequalities of (1) for each z ∈ D we have

$$\Rightarrow -|z|(1+|f(z)|) \leq |f(z)|-1$$

$$\Rightarrow 1-|z| \leq |f(z)|(1+|z|)$$

$$\Rightarrow \frac{1-|z|}{1+|z|} \leq |f(z)| \ \ z \in \mathbb{D} \qquad (2)$$

By taking last two inequalities of (1) for each z ∈ D we have

$$\Rightarrow |f(z)|-1 \leq |z|(1+|f(z)|)$$

$$\Rightarrow |f(z)|(1-|z|) \leq 1+|z|$$

$$\Rightarrow |f(z)| \leq \frac{1+|z|}{1-|z|} \ \ z \in \mathbb{D} \qquad (3)$$

Frpm (2) and (3) we have

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|} \ \ z \in \mathbb{D}$$

3. Conclusion

In this paper, we conclude that modulus of any analytic function on the open unit disk always lies in the annulus of two circles which has various applications to study the range of various type of analytic functions on the disks of large radius. This result also helps to prove the some typical results in the complex analysis like Little Picard Theorem and Bloch's theorem.

4. References

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2. John B Conway. Functions of one complex variable II, volume 159. Springer Science & Business Media, 2012.