International Journal of Statistics and Applied Mathematics 2020; 5(3): 154-158

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2020; 5(3): 154-158

www.mathsjournal.com
Received: 01-03-2020
Accepted: 05-04-2020

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On the maximum real part of an entire function represented by multiple Dirichlet series

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Abstract
Here in this paper we prove an inequality between the coefficients $a_{m,n}$ of multiple Dirichlet Series f(s_1, s_2) and the maximum real part A (σ_1, σ_2) of $f(s_1, s_2)$.

Keywords: Maximum real part, entire function, multiple Dirichlet series

Introduction
Let us consider

(1.1) $\quad f(s_1, s_2) = \sum_{m=0}^{\infty} \quad \sum_{n=0}^{\infty} a_{m,n} \exp(s_1\lambda_m + s_2\mu_n),$

$\left((s_j = \sigma_j + it_j), j = 1,2\right)$

Where $a_{m,n} \in C$, the field of complex numbers, $\lambda'_m s, \mu'_n s$ are real, and

$0 < \lambda_1 < \lambda_2 < \cdots \lambda_m \to \infty;$

$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty.$

It has been proved by Janusauskas [1] that if

(1.2) $\lim_{m\to\infty} \frac{\log m}{\lambda_m} = 0, \lim_{n\to\infty} \frac{\log n}{\mu_n} = 0,$

Then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence.
Also, Sarkar [2, pp.99] has shown that the necessary and sufficient condition that the series (1.1) satisfying (1.2) to be entire is that

(1.3) $\lim_{(m,n)\to\infty} \frac{\log|a_{m,n}|}{\lambda_m+\mu_n} = -\infty$

Let the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3) be denoted by F.
Then f $\in F$ denotes an entire function over C^2. The results can be extended to several complex variables.
Corresponding to an f $\in F$, the maximum real part on R^2 is defined as
A $(\sigma_1, \sigma_2; f)$ = max { $|Re\, f(s_1, s_2)|: s_1, s_2 \in C, Re\, s_1 = \sigma_1, Re\, s_2 = \sigma_2\}$

1. Theorem 1: Let f $\in F$ *be an entire function. Then*

$|a_{m,n}|$ Exp $(\lambda_m\sigma_1 + \mu_n\sigma_2) \leq 4$ A $(\sigma_1, \sigma_2) - 2\, a'_{0,0},$

for all positive values of m, n and $\sigma > (\sigma_1^0, \sigma_2^0)$

Proof
Writing

$s_1 = \sigma_1 + it_1, s_2 = \sigma_2 + it_2, a_{m,n} = a'_{m,n} + ia''_{m,n},$

$f(s_1, s_2) = \phi(\sigma_1, \sigma_2, t_1, t_2) + \text{i} \ (\sigma_1, \sigma_2, t_1, t_2),$
We have

$\phi(\sigma_1, \sigma_2, t_1, t_2) + \text{i} \ (\sigma_1, \sigma_2, t_1, t_2) =$

$a_{0,0} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a'_{m,n} + i\, a''_{m,n}) \exp\{(\sigma_1 + it_1)\lambda_m + (\sigma_2 + it_2)\mu_n\}$

$= a_{0,0} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a'_{m,n} + i\, a''_{m,n}) \exp(\lambda_m\sigma_1 + \mu_n\sigma_2) \exp\{\text{i}(\lambda_m t_1 +$

$\mu_n t_2)\}$

$= a_{0,0} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a'_{m,n} + i\, a''_{m,n}) \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$ x

x $\{\cos(\lambda_m t_1 + \mu_n t_2) + \text{i} \sin(\lambda_m t_1 + \mu_n t_2)\}$

Therefore,

$\phi(\sigma_1, \sigma_2, t_1, t_2) = a'_{0,0} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$ x

x $\{ a'_{m,n} \cos(\lambda_m t_1 + \mu_n t_2) - a''_{m,n} \sin(\lambda_m t_1 + \mu_n t_2)\}$

(2.1) $= a'_{0,0} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$ x

x $\{ a'_{m,n} (\cos\lambda_m t_1 \cos\mu_n t_2 - \sin\lambda_m t_1 \sin\mu_n t_2)$

$a''_{m,n} (\sin\lambda_m t_1 \cos\mu_n t_2 + \cos\lambda_m t_1 \sin\mu_n t_2)\}$

The series converges uniformly with respect to t_1, t_2. Hence we may multiply by $\cos\lambda_m t_1 \cos\mu_n t_2$ or $\sin\lambda_m t_1 \sin\mu_n t_2$ and integrate term by term; and we obtain,

$\frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi(\sigma_1, \sigma_2, t_1, t_2) \cos\lambda_m t_1 \cos\mu_n t_2 \, \text{d}t_1 \, \text{d}t_2$

$= 4 \frac{\sin\lambda_m t_1 \sin\mu_n t_2}{\lambda_m \mu_n T_1 T_2} a'_{0,0}$

$+ \frac{a'_{m,n}}{T_1 T_2} (T_1 + \frac{\sin 2\lambda_m T_1}{2\lambda_m}) (T_2 + \frac{\sin 2\mu_n T_2}{2\mu_n}) \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

$+ \sum_{M=1}^{\infty} \sum_{N=1}^{\infty} \frac{a'_{M,N}}{T_1 T_2} \{ \frac{\sin(\lambda_M + \lambda_m) T_1}{(\lambda_M + \lambda_m)} + \frac{\sin(\lambda_M - \lambda_m) T_1}{(\lambda_M - \lambda_m)}\}$ x

x $\{ \frac{\sin(\mu_N + \mu_n) T_2}{(\mu_N + \mu_n)} + \frac{\sin(\mu_N - \mu_n) T_2}{(\mu_N - \mu_n)}\} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

$= 4 \frac{\sin\lambda_m T_1 \sin\mu_n T_2}{\lambda_m \mu_n T_1 T_2} a'_{0,0} + a'_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$ x

x $(1 + \frac{\sin 2\lambda_m T_1}{2\lambda_m T_1}) (1 + \frac{\sin 2\mu_n T_2}{2\mu_n T_2}) +$

$+ \sum_{M=1}^{\infty} \sum_{N=1}^{\infty} a'_{M,N} \exp(\lambda_M\sigma_1 + \mu_N\sigma_2) \{ \frac{\sin(\lambda_M + \lambda_m) T_1}{(\lambda_M + \lambda_m) T_1} + \frac{\sin(\lambda_M - \lambda_m) T_1}{(\lambda_M - \lambda_m) T_1}\}$ x

x $\{ \frac{\sin(\mu_N + \mu_n) T_2}{(\mu_N + \mu_n) T_2} + \frac{\sin(\mu_N - \mu_n) T_2}{(\mu_N - \mu_n) T_2}\}$

Therefore,

(2.2) $\lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi\,(\sigma_1, \sigma_2, t_1, t_2)\cos\lambda_m t_1 \cos\mu_n t_2\, dt_1\, dt_2$

$= a'_{m,n} \exp(\lambda_m \sigma_1 + \mu_n \sigma_2)$

Also,

$\frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi\,(\sigma_1, \sigma_2, t_1, t_2)\sin\lambda_m t_1 \sin\mu_n t_2\, dt_1\, dt_2$

$= - a'_{m,n} \exp(\lambda_m \sigma_1 + \mu_n \sigma_2) \{ 1 - \frac{\sin 2\lambda_m T_1}{2\lambda_m T_1} \} \{ 1 - \frac{\sin 2\mu_n T_2}{2\mu_n T_2} \} -$

$\sum_{M=1}^{\infty} \sum_{N=1}^{\infty} a'_{M,N} \exp(\lambda_M \sigma_1 + \mu_N \sigma_2)$ x

x $\{ \frac{\sin(\lambda_M - \lambda_m)T_1}{(\lambda_M - \lambda_m)T_1} - \frac{\sin(\lambda_M + \lambda_m)T_1}{(\lambda_M + \lambda_m)T_1} \}$ x

x $\{ \frac{\sin(\mu_N - \mu_n)T_2}{(\mu_N - \mu_n)T_2} - \frac{\sin(\mu_N + \mu_n)T_2}{(\mu_N + \mu_n)T_2} \}$

Hence

(2.3) $\lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi\,(\sigma_1, \sigma_2, t_1, t_2)\sin\lambda_m t_1 \sin\mu_n t_2\, dt_1\, dt_2$

$= - a'_{m,n} \exp(\lambda_m \sigma_1 + \mu_n \sigma_2)$

Similarly multiplying (2.1) by $\mathbf{\sin\lambda_m t_1 \cos\mu_n t_2}$ *or* $\mathbf{\cos\lambda_m t_1 \sin\mu_n t_2}$**,** we obtain

$\frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi\,(\sigma_1, \sigma_2, t_1, t_2)\sin\lambda_m t_1 \cos\mu_n t_2\, dt_1\, dt_2$

$= - a''_{m,n} \exp(\lambda_m \sigma_1 + \mu_n \sigma_2) \{ 1 - \frac{\sin 2\lambda_m T_1}{2\lambda_m T_1} \} \{ 1 + \frac{\sin 2\mu_n T_2}{2\mu_n T_2} \} -$

$\sum_{M=1}^{\infty} \sum_{N=1}^{\infty} a''_{M,N} \exp(\lambda_M \sigma_1 + \mu_N \sigma_2)$ x

x $\{ \frac{\sin(\lambda_M - \lambda_m)T_1}{(\lambda_M - \lambda_m)T_1} - \frac{\sin(\lambda_M + \lambda_m)T_1}{(\lambda_M + \lambda_m)T_1} \}$ x

x $\{ \frac{\sin(\mu_N + \mu_n)T_2}{(\mu_N + \mu_n)T_2} - \frac{\sin(\mu_N - \mu_n)T_2}{(\mu_N - \mu_n)T_2} \}$

Therefore,

(2.4) $\lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi\,(\sigma_1, \sigma_2, t_1, t_2)\sin\lambda_m t_1 \cos\mu_n t_2\, dt_1\, dt_2$

$= - a''_{m,n} \exp(\lambda_m \sigma_1 + \mu_n \sigma_2)$

Also,

$\frac{1}{T_1 T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \phi\,(\sigma_1, \sigma_2, t_1, t_2)\cos\lambda_m t_1 \sin\mu_n t_2\, dt_1\, dt_2$

$= - a''_{m,n} \exp(\lambda_m \sigma_1 + \mu_n \sigma_2) \{ 1 + \frac{\sin 2\lambda_m T_1}{2\lambda_m T_1} \} \{ 1 - \frac{\sin 2\mu_n T_2}{2\mu_n T_2} \} -$

$\sum_{M=1}^{\infty} \sum_{N=1}^{\infty} a''_{M,N} \exp(\lambda_M \sigma_1 + \mu_N \sigma_2)$ x

x $\{ \frac{\sin(\lambda_M + \lambda_m)T_1}{(\lambda_M + \lambda_m)T_1} + \frac{\sin(\lambda_M - \lambda_m)T_1}{(\lambda_M - \lambda_m)T_1} \}$ x

x $\{ \frac{\sin(\mu_N - \mu_n)T_2}{(\mu_N - \mu_n)T_2} - \frac{\sin(\mu_N + \mu_n)T_2}{(\mu_N + \mu_n)T_2} \}$

Hence,

(2.5) $\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)\cos\lambda_m t_1 \sin\mu_n t_2\, dt_1\, dt_2$

$= - a''_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

From (2.2) and (2.3), we get

$\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)$ x

x $(\cos\lambda_m t_1\cos\mu_n t_2 - \sin\lambda_m t_1\sin\mu_n t_2)\, dt_1\, dt_2$

$= 2\, a'_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

Or

(2.6) $\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)\cos(\lambda_m t_1 + \mu_n t_2)\, dt_1\, dt_2$

$= 2\, a'_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

Also from (2.4) and (2.5), we get

$\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)$ x

x $(\sin\lambda_m t_1\cos\mu_n t_2 - \cos\lambda_m t_1\sin\mu_n t_2)\, dt_1\, dt_2$

$= - 2\, a''_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

Or

(2.7) (-) $\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)\sin(\lambda_m t_1 + \mu_n t_2)\, dt_1\, dt_2$

$= 2\, a''_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

Multiplying (2.7) by i and adding to (2.6), we have

$\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)\exp\{- i(\lambda_m t_1 + \mu_n t_2)\, dt_1\, dt_2$

$= 2\, (a'_{m,n} + a''_{m,n}) \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

$= 2\, a_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

Hence,

(2.8) $2\,|a_{m,n}| \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)$

$\leq \lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} |\phi(\sigma_1,\sigma_2,t_1,t_2)|\, dt_1\, dt_2$

Also we have from (2.1),

$\frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} |\phi(\sigma_1,\sigma_2,t_1,t_2)|\, dt_1\, dt_2$

$= 4\, a'_{0,0} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 4\, a'_{m,n} \exp(\lambda_m\sigma_1 + \mu_n\sigma_2)\frac{\sin\lambda_m T_1 \sin\mu_n T_2}{\lambda_m\mu_n T_1 T_2}$

Therefore,

(2.9) $\lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \phi(\sigma_1,\sigma_2,t_1,t_2)\, dt_1\, dt_2 = 4\, a'_{0,0}$

Adding (2.8) and (2.9), we get

$2\,|a_{m,n}| \exp(\lambda_m\sigma_1 + \mu_n\sigma_2) + 4\, a'_{0,0}$

$\leq \lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \{|\phi(\sigma_1,\sigma_2,t_1,t_2)| + \phi(\sigma_1,\sigma_2,t_1,t_2)\}\, dt_1\, dt_2$

Hence,

$|a_{m,n}|$ exp $(\lambda_m \sigma_1 + \mu_n \sigma_2) \leq 4$ A $(\sigma_{1,} \sigma_2) - 2\, a'_{0,0}$, for $\sigma > (\sigma_1^0, \sigma_2^0)$

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