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Parameter estimation of area biased Rayleigh distribution

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Abstract
Area biased Rayleigh distribution is considered. The classical maximum likelihood estimator has been obtained. Bayesian method of estimation is employed in order to estimate the parameter of area biased Rayleigh distribution by using quasi and inverted gamma priors. In this paper, the Bayes estimators of the parameter have been obtained under squared error, precautionary and weighted loss functions.

Keywords: Area biased Rayleigh distribution, Bayesian method, quasi and inverted gamma priors, squared error, precautionary and weighted loss functions

1. Introduction

A weighted form of Rayleigh distribution has been published by Reshi *et al.* [1]. They introduced a new class of size-biased generalized Rayleigh distribution. A weighted model based on the Rayleigh distribution is proposed by Ajami and Jahanshahi [2]. The probability density function of the area biased Rayleigh distribution is given by

$$f(x;\theta) = \frac{1}{2}\theta^{-2}x^3 e^{-x^2/2\theta} \quad ; x>0,\ \theta>0. \tag{1}$$

The joint density function or likelihood function of (1) is given by

$$f(\underline{x};\theta) = \left(\frac{1}{2}\right)^n \theta^{-2n}\left(\prod_{i=1}^{n} x_i^3\right) e^{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2} \tag{2}$$

The log likelihood function is given by

$$\log f(\underline{x};\theta) = n\log\left(\frac{1}{2}\right) - 2n\log\theta + \log\left(\prod_{i=1}^{n} x_i^3\right) - \frac{1}{2\theta}\sum_{i=1}^{n} x_i^2 \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get

$$\hat{\theta} = \frac{1}{4n}\sum_{i=1}^{n} x_i^2 \tag{4}$$

2. Bayesian Method of Estimation

In Bayesian analysis the fundamental problem are that of the choice of prior distribution g (θ) and a loss function $L\left(\hat{\theta},\theta\right)$. The squared error loss function for the scale parameter θ are defined as

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2 \qquad (5)$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E\left(\theta\right) \qquad (6)$$

Zellner [3], Basu and Ebrahimi [4] have recognized that the inappropriateness of using symmetric loss function. Norstrom [5] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}} \qquad (7)$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation.

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2} \qquad (8)$$

Weighted loss function (Ahamad *et al.*) [6] is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\theta} \quad . \qquad (9)$$

The Bayes estimator under weighted loss function is denoted by $\hat{\theta}_W$ and is obtained as

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} \qquad (10)$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.
(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1\left(\theta\right)=\frac{1}{\theta^d} \;;\; \theta>0,\; d\ge 0, \qquad (11)$$

Where
$d = 0$ leads to a diffuse prior and
$d = 1$, a non-informative prior.
(ii) Inverted gamma prior: The most widely used prior distribution of θ is the gamma distribution with parameters α and $\beta\left(>0\right)$ with probability density function given by

$$g_2\left(\theta\right)=\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}\theta^{-(\alpha+1)}e^{-\beta/\theta} \;;\; \theta>0. \qquad (12)$$

3. Bayes Estimators under $g_1\left(\theta\right)$

The posterior density of θ under $g_1\left(\theta\right)$, on using (2), is given by

$$f\left(\theta/\underline{x}\right)=\frac{\left(\frac{1}{2}\right)^{n}\theta^{-2n}\left(\prod_{i=1}^{n}x_i^3\right)e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2}\theta^{-d}}{\int_0^\infty\left(\frac{1}{2}\right)^{n}\theta^{-2n}\left(\prod_{i=1}^{n}x_i^3\right)e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2}\theta^{-d}\,d\theta}$$

$$=\frac{\theta^{-(2n+d)}\,e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2}}{\int_0^\infty\theta^{-(2n+d)}\,e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2}\,d\theta}$$

$$=\frac{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)}\,\theta^{-(2n+d)}\,e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2} \tag{13}$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_S=\frac{\frac{1}{2}\sum_{i=1}^{n}x_i^2}{2n+d-2} \tag{14}$$

Proof. From equation (6), on using (13),

$$\hat{\theta}_S=E(\theta)=\int\theta f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)}\int_0^\infty\theta^{-(2n+d-1)}\,e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2}\,d\theta=\frac{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)}\frac{\Gamma(2n+d-2)}{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-2}}\ \text{Or,}\ \hat{\theta}_S=\frac{\frac{1}{2}\sum_{i=1}^{n}x_i^2}{2n+d-2}$$

Theorem 2. Assuming the precautionary loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_P=\frac{\frac{1}{2}\sum_{i=1}^{n}x_i^2}{\left[(2n+d-2)(2n+d-3)\right]^{\frac{1}{2}}} \tag{15}$$

Proof. From equation (8), on using (13),

$$\left(\hat{\theta}_P\right)^2=E\left(\theta^2\right)=\int\theta^2 f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)}\int_0^\infty\theta^{-(2n+d-2)}\,e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i^2}\,d\theta=\frac{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)}\frac{\Gamma(2n+d-3)}{\left(\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+d-3}}$$

$$= \frac{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^2}{(2n+d-2)(2n+d-3)}$$

$$\Rightarrow \qquad \hat{\theta}_P = \frac{\frac{1}{2}\sum_{i=1}^{n} x_i^2}{\left[(2n+d-2)(2n+d-3)\right]^{\frac{1}{2}}}$$

Theorem 3. Assuming the weighted loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_W = \frac{\frac{1}{2}\sum_{i=1}^{n} x_i^2}{2n+d-1} \qquad (16)$$

Proof. From equation (10), on using (13),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f(\theta/\underline{x})\, d\theta\right]^{-1}$$

$$= \left[\frac{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)} \int_0^\infty \theta^{-(2n+d+1)}\, e^{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2}\, d\theta\right]^{-1}$$

$$= \left[\frac{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+d-1}}{\Gamma(2n+d-1)} \frac{\Gamma(2n+d)}{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+d}}\right]^{-1} = \left[\frac{2n+d-1}{\frac{1}{2}\sum_{i=1}^{n} x_i^2}\right]^{-1}$$

or, $\hat{\theta}_W = \dfrac{\frac{1}{2}\sum_{i=1}^{n} x_i^2}{2n+d-1}$

4. Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f(\theta/\underline{x}) = \frac{\left(\frac{1}{2}\right)^n \theta^{-2n} \left(\prod_{i=1}^{n} x_i^3\right) e^{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}}{\int_0^\infty \left(\frac{1}{2}\right)^n \theta^{-2n} \left(\prod_{i=1}^{n} x_i^3\right) e^{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}\, d\theta}$$

$$= \frac{\theta^{-(2n+\alpha+1)}\, e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)}}{\int_0^\infty \theta^{-(2n+\alpha+1)}\, e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)} d\theta}$$

$$= \frac{\theta^{-(2n+\alpha+1)}\ e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)}}{\Gamma(2n+\alpha)\Big/\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+\alpha}}$$

$$= \frac{\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)}\theta^{-(2n+\alpha+1)}\ e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)} \qquad (17)$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_S = \frac{\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2}{2n+\alpha-1} \qquad (18)$$

Proof. From equation (6), on using (17),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)}\int_0^\infty \theta^{-(2n+\alpha)}\ e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)} d\theta$$

$$= \frac{\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)}\frac{\Gamma(2n+\alpha-1)}{\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+\alpha-1}}$$

or, $$\hat{\theta}_S = \frac{\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2}{2n+\alpha-1}.$$

Theorem 5. Assuming the precautionary loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_P = \frac{\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2}{\left[(2n+\alpha-1)(2n+\alpha-2)\right]^{\frac{1}{2}}} \qquad (19)$$

Proof. From equation (8), on using (17),

$$\left(\hat{\theta}_P\right)^2 = E(\theta^2) = \int \theta^2 f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)}\int_0^\infty \theta^{-(2n+\alpha-1)}\ e^{-\frac{1}{\theta}\left(\beta+\frac{1}{2}\sum_{i=1}^{n}x_i^2\right)} d\theta$$

$$= \frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)} \frac{\Gamma(2n+\alpha-2)}{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+\alpha-2}}$$

$$= \frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2}}{(2n+\alpha-1)(2n+\alpha-2)}$$

or, $$\hat{\theta}_P = \frac{\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2}{\left[(2n+\alpha-1)(2n+\alpha-2)\right]^{\frac{1}{2}}}.$$

Theorem 6. Assuming the weighted loss function, the Bayes estimate of the parameter θ, is of the form

$$\hat{\theta}_W = \frac{\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2}{2n+\alpha} \tag{20}$$

Proof. From equation (10), on using (17),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f\left(\theta/\underline{x}\right) d\theta\right]^{-1}$$

$$= \left[\frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)} \int_0^\infty \theta^{-(2n+\alpha+2)}\, e^{-\frac{1}{\theta}\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)} d\theta\right]^{-1}$$

$$= \left[\frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+\alpha}}{\Gamma(2n+\alpha)} \frac{\Gamma(2n+\alpha+1)}{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{2n+\alpha+1}}\right]^{-1}$$

$$= \left[\frac{2n+\alpha}{\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2}\right]^{-1}$$

or, $$\hat{\theta}_W = \frac{\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2}{2n+\alpha}.$$

5. Conclusion
In this paper, we have obtained a number of estimators of the parameter of area biased Rayleigh distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (14), (15) and (16) we have obtained the Bayes estimators under squared error, precautionary and weighted loss functions using quasi prior. In equation (18), (19) and (20) we have obtained the Bayes estimators under squared error, precautionary and weighted loss functions using inverted gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

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