International Journal of Statistics and Applied Mathematics 2020; 5(4): 22-26

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2020; 5(4): 22-26

www.mathsjournal.com
Received: 10-05-2020
Accepted: 12-06-2020

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Hermitian matrix inequalities and a conjecture

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Abstract
In many ways Hermitian matrices resemble real numbers. Indeed, all eigenvalues of a Hermitian matrix are real and the matrix is diagonalizable. This similitude may lead an unwary mind to wrong conclusions. This is especially true in the study of inequalities involving Hermitian matrices.
In the sequel we use capital letters A, B,..., X, etc., to denote *n* X *n* Hermitian matrices where *n* is some integer greater than l; *A* = *A** where *A** denotes the conjugate of the transpose of *A.* We use *u* and *v* to denote complex column vectors in C" furnished with the usual inner product (u, *v).* We define

Keywords: Hermitian, resemble, eigenvalues, diagonalizalable, similitude, especially, furnished

Introduction

1. $A \geq (>)0$ if all eigenvalues of A are nonnegative (positive) or equivalently if $(u, Au) \geq (>)0$ for all nonzero vectors $u \in C^n$.
2. $A \geq (>)B$ if $A - B \geq (>)0$, or equivalently if $(u,Au) \geq (>) (u,Bu)$ for all nonzero vectors $u \in C^n$.

Unfortunately this ordering is only a partial one. Thus it is not true that if *A* is not greater than or equal to *B* then *A* must be smaller than *B.* Another source of trouble is the fact that matrix multiplication is not compatible with the ordering (unless all matrices involved are mutually commutative). Most inequalities involving multiplications of real numbers cease to hold when the real numbers are replaced by Hermitian matrices. Two simple examples are:

1. It is not true that $A \geq 0$ and $B \geq 0$ imply that $AB + BA \geq 0$. Simply take $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and B $= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

2. It is not true that $A \geq \quad B \geq 0$ implies that $A^2 \geq B^2$. Take $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and B $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

To illustrate the delicacy of Hermitian inequalities further, we note two more examples:
3. Let *A* and *B* be nonnegative Hermitian such that $A + B > 0$, and let *X* be an *n* X *n* Hermitian matrix that satisfies the inequality

(1.1) $(A+ B) X + X(A + B) \geq AB+ BA.$

Inthe scalar case ($n = 1$), it is obvious that the real number *X* must be positive. However, that is not always true for *n*;,. 2. A counterexample is

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $(A + B)X + X(A + B) = AB+ BA$ but $X \geq 0$.
However, if in addition $A > 0$ and instead of (1.1) *X* satisfies

(1.2) $(A + B)X + X(A + B) \geqslant A^{-1}B + BA^{-1},$

Then $X \geq 0$. In fact if we have equality in (1.2), then *X is* given by $X = A^{-1} - (A+ B)^{-1} \geq 0$.

pair *(A, P)* for which the solution X is not nonnegative definite. For each dimension *n*, there is a cutoff value *t0* such that the result holds for $t \in (-2, t0)$ but not for $t > t0$. It is also known that $t0 > 2$ for all *n*. The question of the exact value of *t0* for $n \geq 4$ is open.

Those few results concerning inequalities that continue to hold are usually hard to prove. The following is a natural question to ask: What functions preserve the ordering of Hennitian matrices? In other words, what must *f* satisfy so that

$$A \geq B \Rightarrow f(A) \geq f(B)?$$

Let us recall how functions of a Hermitian matrix are defined. If *D* is a diagonal matrix with diagonal elements { d1, d 2,...,*dn}, a< d, < b*, and *f*: $(a,b) \rightarrow (-\infty,\infty)$ is any real-valued function, then *f(D)* is defined to be the diagonal matrix with diagonal elements { *f* (d1), *f* (d_2),..,*f(dn)}*. In general, if *A* is not diagonal, there exist a unitary matrix *U* and a diagonal matrix *D* such that $A = U^*DU$. We then define

$f(A)= U^*f(D)U$.

The question was studied and completely resolved by C. Loewner in his 1934 paper [12]. It turns out that the condition on/ depends on the size of the matrices in question. More precisely,

Let $P_n(a, b) = \{f: (a, b) \rightarrow (-\infty, \infty): f(A) \geq f(B)$ for all pairs of *n* X *n* Hermitian matrices *A, B* with eigenvalues in *(a, b)* such that $A \geq B\}$.

Then

$$P_{1,}(a\ b) \nsupseteq P_2\,(a\ b) \nsupseteq \cdots \nsupseteq P_n\,(a\ b) \nsupseteq \cdots$$

The class of functions P_n (a, b) can be characterized by the positivity of a certain matrix formed with divided differences of the function with respect to *n* arbitrary distinct points taken in *(a, b)*.

It is necessary that $f \in P_n\,(a, b)$ possess continuous derivatives up to order 2 *n* - 3 and that $f^{(2n-3)}$ be convex.

Loewner went on to show that the set $P(a, b) = \cap_{n=2}^{\infty} P_n(a, b)$ is miraculously tied up with the theory of analytic functions. Indeed *P(a, b)* is precisely the set of real-valued functions on *(a, b)* that are continuable analytically to the upper half plane in which the functions take values with positive imaginary parts (the so-called Pick functions).

For an exposition of this beautiful theory, we refer the readers to the monograph by Donoghue [6], and the article by Ando [1].

From an advanced point of view, Loewner's result is elegant, but the proof is by no means short or elementary. For instance, it is not at all obvious that $f(t) = \sqrt{t}$ belongs to $P(0, \infty)$. Perhaps this is the reason why the fact "$t^r \in P(0, \infty)$, $0 < r < 1$" has been rediscovered more than once. We shall come back to this in the next section.

It is an interesting exercise to try to discover as many functions in *P(a, b)* as possible by entirely elementary means. In Section 2 we look at this problem. In Sections 3 and 4 we discuss a conjecture and some results that are inspired by an attempt to look for further inequalities.

2. An inequality-generating theorem. We shall establish the following general result.

Theorem 1: Let A, B, C, D be n X n Hermitian matrices. Suppose that A commutes with C and that B commutes with D. If $A \geq B \geq 0$ and $C \geq D \geq 0$, then for any positive r and s such that $r + s \leq 1$ we have

$$A^rC^s \geq B^rD^s$$

The proof of Theorem 1 depends on the following very special case. (Take $r = 1/2$, $s = 0$, and $C = D = I$, the identity matrix).

Theorem 2: *If* $A \geq B \geq 0$, *then* $A^{1/2} \geq B^{1/2}$.

As pointed out in the introduction, Theorem 2 is an obvious corollary of Loewner's deep result.

R. Bellman [3] proved Theorem 2 analytically using ideas in the theory of differential equations.

Y. H. Au-Yeung [2] gave a simple algebraic proof of the following generalization.

Theorem 3: *If* $A \geq B \geq 0$, *then* $A^r \geq B^r$ *for all* $r \in [0, 1]$.

Special cases of Theorem 1 were used by Au-Yeung in his proof of Theorem 3 but Theorem 1 has not been stated in such generality.

Theorem 3 was in fact "discovered" earlier in 1951 by E. Heinz [7], not just for matrices but for general Hermitian operators in a Hilbert space. In the following year T. Kato [8] gave a shorter proof. These authors seem not to have been aware of Loewner's result at the time the papers were written. The various proofs are, however, different and are worth recording. In [9] the second author gave yet another proof that resulted in a stronger form of

Theorem 3. Recently the operator version of Theorem 2 found an interesting application in linear neutron transport theory; see [11]. An important normequivalence result can now be given a simple elementary proof. This is an impetus for further investigation of related inequalities. Still more recently Theorem 2 was used in a study of the oscillation of second order systems of differential equations of the form $X''(t) + Q(t)X(t) = 0$, where *Q(t)* is an *n* X *n* Hermitian matrix for all *t* and *X(t)* is an n-vector, see [5].

The (ollowing proof of Theorem 2 is a simplification of a proof that appeared in Marshall and Olkin [13]. Alternative proofs by other authors can also be found in [13].

Proof of Theorem 2: If suffices to establish the conclusion under the additional condition that

$A > 0$, since the general case follows by continuity. Let

$$C = A^{1/2},\ D = B^{1/2} \text{ and } \quad X = C - D.$$

Then

$$o \leq C^2 - D^2 = C^2-(X- C)^2 = CX+ XC - X^2.$$

Theorem 1 is thus a consequence of the following well-known result due to Lyapunov.

Lemma 1: Suppose C and P are square matrices of the same size with $C > 0$. The equation

(2.1) $\quad CX + XC = P$

Has a unique solution X. Moreover if $P \geq 0$, then so is X

Proof. For the sake of completeness we give the proof. That (2.1) has a unique solution *X* can be checked directly after we notice that we can assume without loss of generality that *C* is diagonal. In particular, by taking adjoints of both sides of (2.1), we conclude that *X* must be Hermitian whenever *P* is. Suppose finally that $P \geq 0$ and *u* is an eigenvector of *X* corresponding to the eigenvalue α.

Then
$0 \leq (u,Pu) = (u,CXu) + (u,XCu) = 2\alpha(u,Cu)$
It follows that $\alpha \geq 0$ and hence X is nonnegative.
An alternative proof is to make use of the explicit formula
$X = \int_0^\infty e^{-Ct} P e^{-Ct}\, dt$ which can be found in Bellman [4, p. 179].
The following lemmas can be found in [2, 9].

Lemma 2: Suppose C≥ 0. Then A ≥ B implies that CAC ≥ CBC. If furthermore C > 0, then the converse also holds.

Proof: It is not difficult to prove the lemma using the definition of ordering via the inner product.

Lemma 3: *If* $A \geq B \geq 0$, *then* B- 1 ≥ A- 1.

Proof: Let I denote the identity matrix. First notice that $X \geq I$ implies that x-1 ≤ I. This can be verified either directly from the definition of x-1 or from Lemma 2 with $A = X$, $B = I$, and $C = X^{-\frac{1}{2}}$ Now Lemma 2 gives $B^{-\frac{1}{2}} AB^{-\frac{1}{2}} \geq I$ from the hypothesis that $A \geq B$. Taking the inverse we get $B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \leq I$. Then the conclusion follows after using Lemma 2 one more time.

Lemma 4: Suppose that G ≥ F > 0 and X ≥ 0, Y ≥ 0. If YFY ≥ XGX, then Y ≥ X.

Proof: By Lemma 2, $YGY \geq YFY$, which together with the second hypothesis gives $YGY \geq XGX$. It follows from Lemma 2 that

$$G^{\frac{1}{2}}YGYG^{\frac{1}{2}} \geq G^{\frac{1}{2}}XGXG^{\frac{1}{2}} \text{ or } \left(G^{\frac{1}{2}}YG^{\frac{1}{2}}\right)^2 \geq \left(G^{\frac{1}{2}}XG^{\frac{1}{2}}\right)^2$$

By Theorem 2, $G^{\frac{1}{2}}YG^{\frac{1}{2}} \geq G^{\frac{1}{2}}XG^{\frac{1}{2}}$. Lemma 2 then gives the conclusion.
We are now ready to give a proof of Theorem 1.
Proof of Theorem l. As in the proof of Theorem 2, we may assume without loss of generality that $B > 0$. By Lemma 3, $B^{-1} \geq A^{-1}$.
Using Lemma 4 with X= $(BD)^{1/2}$, Y= $(AC)^{1/2}$, F = A^{-1} and G = B^{-1}, we obtain the conclusion of the Theorem for the special case r = l/2 and s = 1 - r.
Replacing A and B by (AC)*1/2* and $(BD)^{1/2}$, respectively, in the special case r = l/2 of the Theorem, we obtain the special case r = l/4, s = 3/4.
Replacing C and D by (AC)1/2 and *(BD)1/2*, respectively, in the special case r = l/2, we obtain the special case r = 3/4, s = 1/4. Repeating this process, we see that the Theorem holds when r is any dyadic fraction, namely, when r is of the form $k/2^n$ (n = l, 2,... ; k = l, 2,..., 2^n - 1) and s = 1 - r. Since such fractions are dense in [0,1], the Theorem holds for all general $r \in (0, 1)$ and s = 1 - r by continuity. For s < 1 - r, the required inequality can be written as

$$(A^{r'}B^{s'})^{r+s} \geq (B^{r'}D^{s'})^{r+s},$$

Where r' = r/(r + s), s' = s/(r +s) and r' + s' = 1
We can thus complete the proof of Theorem 1 by using the result established above and Theorem 3, which as has been pointed out is a special case of Theorem 1 when $C = D = I$, $r + s = 1$.
An immediate consequence of Theorem 1 is the following property of the Pick functions. Let *P0* (0, ∞) be the subfamily {f∈ P(0,∞): f: (0,∞) → [0,∞)} of P(0,∞).

Theorem 4: For any r, s > 0 such that r + s ≤ l and any f, g ∈ P_0 (0,∞), we have f^r, $g^s \in P_0$ (0,∞), More generally, let $r_1 > 0$ (i = 1,2,...,n) be such that $\sum_{k=1}^{m} r_1 \leq 1$.
Then

$\prod_{i=1}^{m} f_i^{r_1} \in P_0(0,\infty)$ *for all* f_i ∈ P0 (0, ∞) (i = 1,2,….,n).
In other words, the set of functions {Inf: f ∈ *P0 (0, ∞)*} is convex.
Let us show how further examples of functions in *P0 (0, oo)* can be generated. We assume throughout the hypotheses $A \geq B \geq 0$; in a few cases the stronger condition $B \geq 0$ is needed. Notice that in any case we need only establish the result under the stronger condition $B > 0$ as the general case follows by continuity. Also notice that Theorem 4 can be used in the obvious way to yield generalizations.
Let r, s > 0, $r + s \leq$ l and $\alpha, \beta > 0$. Then a simple consequence of Theorem 4 is
(1) $(\alpha I + A)^r(\beta I + A)^s \geq (\alpha I + B)^r(\beta I + B)^s$
1. For all $a \geq \gamma > 0$, $0 \leq r \leq 1$.
(2) $(\gamma I + A)(\alpha I + A)^{-r} \geq (\gamma I + B)^r(\alpha I + B)^{-r}$
2. or equivalently for all $\beta > 0, \beta^{-1} \geq \gamma$
(3) $(\gamma I + A)(I + \beta A)^{-r} \geq (\gamma I + B)(I + \beta B)^{-r}$
Indeed by Lemma 3, $A^{-1} \leq B^{-1}$. Thus $(I + \alpha A^{-1}) \leq (I + B^{-1})$.
Lemma 3 again yields
$(I + \alpha A^{-1})^{-1} \geq (I + B^{-1})^{-1}$. which is exactly (2) with r = l, γ = 0. Inequality (2) with γ = 0
now follows if in Theorem 1 we let $C = A(\alpha I + A)^{-1}$ and $D = B(\alpha I + B)^{-1}$. To see that (2) holds in the general case $\gamma \neq 0$ we replace A by $\gamma I + A$ and α by $\alpha - \gamma$.
Integrating a known inequality with respect to some parameter leads to another inequality. The following are three examples.
(4) $A[\ln(I+A)]^{-r} \geq B[\ln(I+B)]^{-r}$

for all $r \in [0, 1]$. We require $B > 0$ to guarantee that $[\ln(I + B)]^{-r}$ is defined. However, in the singular case we can still define $B[\ln(I + B)]^{-r}$ by a limiting process, since $lim_{t\to 0}$ $t[\ln(1 + t)]^{-r}$ exists as can be seen using L'Hospital's rule. By

Theorem 3:
$(I+A)^r \geq (I+B)^r$, r∈[0,1].
Integrating this inequality with respect to rover [0,1] gives (4) with r = l. The general case then follows upon applying Theorem 1.
In (2) we let γ = 0 and r = l and integrate the resulting inequality with respect to a over [0,1] and finally invoke Theorem 1 to get
(5) $A[ln(I + A^{-1})]^r \geq B[ln(I + B^{-1})]^r$
On the other hand, if in (3) we let r = l, restrict γ to be in [0,l], and integrate with respect to
β over [0,1], we obtain
(6) $(I + \gamma A^{-1}) \ln(I + A) \geq (I + \gamma B^{-1}) ln(I + B)$.
In particular we have
(7) $\ln(I + A) \geq \ln(I + B)$.
We let the readers investigate for themselves what will result from integrating, for instance,
$rA^r \geq rB^r$, $r \in [0, 1]$, or inequality (2) with γ = 0, r = l and α replaced by $\alpha 2$, $\alpha \in [0, b]$.
In (7) we can replace A by nA and B by nB to deduce
$\ln\left(\frac{1}{n} + A\right) \geq \ln\left(\frac{1}{n} + B\right)$
Letting n → ∞ we get
(8) ln A ≥ ln B.
Here $f(t)$ = ln tis an example of a function belonging to *P(0,* ∞) but not to *P0 (0,* ∞). Inequality (8) Can also be derived from Theorem 3 via differentiation. By Theorem 3,
$A^r - I \geq B^r - I$.
Dividing by r and taking limits as $r \to 0$, we have

$$\frac{dA^r}{dr}\Big|_{r=0} \geq \frac{dB^r}{dr}\Big|_{r=0}$$

Which is (8).

The findings above can be summarized as follows:

Examples of functions in *P0* (0, ∞) include

$(\alpha+t)^r(\beta+t)^s$ $\alpha,\beta>0$ *r,s* > 0, r + s ≤ 1

$(\gamma+t)/(\alpha+t)^r$ $\alpha\ge\gamma>0$ $r\in[0,1]$

$(\gamma+t)/(1+\beta t)^r$ $\beta^{-1}\ge\gamma>0$ $r\in[0,1]$

$t/\ln^r(1+t)$ $r\in[0,1]$

$t\ln^r(1+1/t)$ $r\in[0,1]$

$(1+\gamma/t)\ln(1+t)$ $\gamma\in[0,1]$

$(t\ln t-t+1)\ln^{-2}t$

$\sqrt{t}\ \tan^{-1}(b/\sqrt{t})$ $b\in(0,\infty)$;

ln *t* belongs to *P(0, ∞)*. Both *P0* (0, ∞) and *P (0, ∞)* are closed cones (hence the integration technique for generating new examples works) and furthermore *P0 (0, ∞)* satisfies Theorem 4. It should be stated again that all these can be directly deduced from Loewner's original result.

3. A conjecture. Taking square roots of course restores the inequality that is destroyed after squaring. It is interesting to ask if the following sequence of operations will preserve the inequ ality:

1. Squaring the two sides of the inequality *A* ≥ *B* ≥ 0 to get *(A^2, B^2).*
2. Multiplying both on the right and on the left by some *C* > 0 to get (*CA2C, C B 2C).*
3. Taking square roots

The question is whether

$(CA^2C)^{1/2}\ \ge\ (CB^2C)^{1/2}$.

Lemma 2 seems to indicate that the second operation should not have worsened the situation. Thus one.is tempted to answer yes. However, after some trial and error using the MAT-LAB package on the VAX at Argonne National Laboratory, the following counter example was discovered. Take

$A=\begin{pmatrix}5 & 2\\ 2 & 101\end{pmatrix}$, $B=\begin{pmatrix}1 & 0\\ 0 & 100\end{pmatrix}$, And $C=\begin{pmatrix}1 & 0\\ 0 & 0.2\end{pmatrix}$.

It is easy to check that *A* - *B* ≥ 0. The computer output shows (originally with 16 decimal places).

$$E=(CA^2C)^{1/2}-(CB^2C)^{1/2}=\begin{pmatrix}4.116681\ldots & 1.679158\ldots\\ 1.679158\ldots & 0.134061\ldots\end{pmatrix}$$

The eigenvalues of *E* are -0.4794... and 4.730... Thus $(CA^2C)^{1/2}\ngeq(CB^2C)^{1/2}$

The same experiment seems to indicate that with the special choice *C* = *B* or *A*, the inequality persists. We therefore propose the following conjecture. We have found a proof for the case of the lowest dimension *n* = 2. It is proved by sheer brute force although the proof is by no means str aightforward.

Conjecture: *If A* ≥ *B* ≥ 0, *then*

(3.1) $(BA^2B)^{1/2}\ \ge B^2$, *and* $B^2\ge(AB^2A)^{1/2}$.

Notice that the second inequality is a consequence of the first one. We may assume without loss of generality that *B* > 0 so that B^{-1} exists. By hypothesis, $B^{-1}\ge A^{-1}>0$. The first inequality then gives $(A^{-1}B^{-2}A^{-1})^{1/2}\ge A^{-2}$ Taking inverses now gives (3.2). It is tempting to guess that the similar inequality

$$(AB^2A)^{1/2}\ \ge B^2$$

also follows from the hypotheses of the conjecture. If this were true, then by transitivity (using (3.2) at least for *n* = 2), it would follow that $A^2\ge B^2$, which we know is false.

Repeated use of (3.1) shows that

(3.3) $(B^3A^2B^3)^{1/2}\ \ge B^4$

and more generally

(3.4) $(B^{m-1}A^2B^{m-1})^{1/2}\ \ge B^m$

for *m* = 2^k, where *k* = l, 2, We know that (3.4) is true at least in the two-dimensional case. It is only natural to ask if (3.4) is true for all *m* ∈ (1, ∞) and for all dimension *n*

4. Some evidence for the conjecture. Let us see how Conjecture l leads to another proposition. Let *B* be any positive Hermitian matrix and *P* be any nonnegative matrix. Define

$$Y(t)=[B(B+tP)^2B]^{1/2}$$

By Conjecture 1, *Y(t)*≥ *Y(0) for* t ≥ 0. Thus

(4.1) Y'(0) ≥ 0.

By differentiating the equation *Y 2(t) = B(B + tP)*^{2} *B* and then letting *t* = 0, we see that *X = Y' (O)* satis fies the matrix equation

(4.2)*B 2 X + XB 2 = B 2PB + BPB 2.*

Notice that *Q* = *BPB* is also nonnegative. Thus the validity of the following is a consequence of that of Conjecture 1.

Theorem 5. Let B be a positive Hermitian matrix and Q a nonnegative Hermitian matrix. The solution **X** of the following matrix equation is always nonnegative:

(4. 3) *B 2 X + XB 2 = BQ + QB.*

If the right-hand side is nonnegative, then, by Lemma 1, the conclusion holds. However, as noted before *BQ* + *QB* is in general not nonnegative.

Proof of Theorem 5. Let *Y* be the solution of

(4.4) *B^2Y + Y B^{2} = Q.*

By Lemma 1 of Section 2, *Y* ≥ 0. Thus it follows from Lemma 2 that BYB ≥ 0. Direct computation shows that *BY*+ *YB* satisfies equation (4.3). Hence by uniqueness (Lemma 1)

X = BY + YB.

Now

BX+ XB = B(BY+ YB) +(BY+ YB)B

= Q + 2BYB

≥ 0.

Thus by Lemma 1 again *X* ≥ 0.

COROLLARY 6. *Let* b1,..., b_n,, *be n positive numbers. The determinant*

(4.5) $\det\left(\frac{b_ib_j(b_i^2+b_j^2)}{(b_i^4+b_j^4)}\right)_{i,j=1,\ldots n.}$

is nonnegative.

Proof. Apply Theorem 5 with *B* = diag (b1,..., b_n,*)* and *Q* = $B^{1/2}RB^{1/2}$ where all the entries of *R* are 1- direct computation shows that the solution of (4.3) is given by (4.5).

It was originally planned to derive Theorem 5 as a consequence of Corollary 6 which can be proved directly in case *n* = 2 and 3.

It is interesting to point out that the following generalization of Theorem 5 is still true.

Theorem 7: Lets be any real number in [1, ∞) and B, Q be as in Theorem 5. Then the solution of the matrix equation

(4. 6) $B^sZ+ZB^s=BQ+QB$

is nonnegative.

Direct computation easily confirms the case *n* = 2. Lemma 1 can be thought of as the limiting case s→∞ (with $B^s=C$ and $B=C^{1/s}$→ identity). Notice that the Theorem is false for *s* < l.

Theorem 7 and other generalizations will be given in a forthcoming paper.

A most natural question to ask is: How can one characterize all functions *f* such that the solution of the matrix equation *f(B)Z* + *Zf(B)* = *BQ* + *QB* is nonnegative?

To conclude the paper we mention another tantalizing conjecture.

PROBLEM. Let A, Band C be nonnegative Hermitian matrices such that A ≤ C, B ≤ C. Is it true that

$$(A^2+B^2)^{\frac{1}{2}}\le\sqrt{2}\,C\ ?$$

There are counter examples to the more general conject ure: *A* ≤ C,

$B \leq D$ implies

$$(A^2 + B^2)^{1/2} \leq (C^2 + D^2)^{1/2}.$$

References

1. Ando T. Concavity of certain maps on positive definite matrice s and applications to Hadamard products, Linear Algebra and Appl. 1979; 26:203-241.
2. Au-Yeung YH. Some inequalities for the rational power of a nonn egative definite matrix, Linear Algebra Appl. 1973; 7:347-350.
3. Bellman RE. Some inequaliti es for the square root of a positive definite matrix, Linear Algebra App l., 1 (1968) 321-324.
4. Introduction to Matrix Analysis, McGraw-Hill, New York, 1970.
5. Byers R, Harris BJ, Man Kam Kwong, Weighted means and oscillation conditions for second order matrix differential equations, J. Differential Equations (to appear).
6. Donoghue WF. Monotone Matrix Functions and Analytic Continuation, Springer-Verlag, Berlin, 1974.
7. Heinz E. Beitrage zur Storungsteorie der Spektralzerlegung, Math. Ann. 1951; 123:415-538.
8. Kato T. Notes on some inequalities for linear operators, Math. Ann. 1952; 125:208-212.
9. Man Kam Kwong, Inequalities for the powers of nonnegative Hermitian operators, Proc. Amer. Math. Soc. 1975; 51:401-406.
10. On the definiteness of the solution of certain matrix equation, Linear Algebra and Appl. (to appear).
11. Man Kam Kwong and H. Kaper, preprint.
12. Loewner C. Uber monotone Matrixfunktionen, Math. Z., 1934; 38:177-216.
13. Marshall AW, Olkin I. Inequalities: Theory of Majorization and Its Applications, Academic Press, New York, 1979.