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On numerical range and spectral properties of some classes of operators in hilbert spaces

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Abstract
It is a well known fact in operator theory that the numerical range and spectral properties are significant in determining the behavior of different classes of operators. In this paper we focus mainly on the class of compact operators in Hilbert spaces. AMS subject classification 47B47, 47A30, 47B20.

Keywords: Compact operators, numerical range and spectral properties

1. Introduction
The numerical and spectral properties of operators have been jointly studied by a number of authors among them Toeplitz [4] who showed that spectrum of any operator is contained in the closure of the numerical range of operator. It has also been shown that the convex hull of the spectrum is also contained in the closure of the numerical range. However if the operator is normal then the convex hull equals the closure of the numerical range see also Halmos [2]. For a normal operator the spectrum and closure of the numerical range are spectral sets for the operator. Some authors have made comparison on the properties relating the spectrum of an operator and the closure of its numerical range but in this paper we identify spectral properties and the numerical range of compact operators as we investigate the fact that for such operators, the spectrum is contained in their numerical range.

2. Notations, definitions and terminologies
Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$ the numerical range of A is denoted by $W(A)$ and defined as,
$W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$.
The closure of the numerical range of A is denoted by $\overline{W(A)}$ while the spectrum of A is denoted by $\sigma(A)$ and defined as $\sigma(A) = \{\lambda \in \mathbb{C}: A - \lambda I \text{ } is\ not\ invertible\}$ and the convex hull of the spectrum of A is denoted by Conv $\sigma(A)$.
Given a complex Hilbert space $\mathcal{H}$ and operators $A, B \in \mathcal{B}(\mathcal{H})$ the commutator of A and B is given by $[A, B] = AB - BA$.
The point spectrum of A is given by $\sigma_p(A) = \{\lambda \in \mathbb{C}: Ax = \lambda x\}$. Thus $\sigma_p(A) \subseteq \sigma(A) \subseteq \overline{W(A)}$,
also Conv $\sigma(A) \subseteq \overline{W(A)}$.
We also denote range of A and kernel of A by ran A and Ker A respectively and a closed subset X of the complex plane $\mathbb{C}$ is a spectral set for an operator, $A \in \mathcal{B}(\mathcal{H})$ if:
(i) $\sigma(A) \subset X$ (ii) $\|f(A)\| \leq \|f\|_X = sup\{|f(z)|: z \in X\}$ .For all $f \in R(X)$ where $R(X)$ is the algebra of all complex valued rational functions whose poles are not in X
Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be:
- Self adjoint if A= A^* $(A^*\ is\ adjoint\ of\ opertor\ \ A)$
- Normal if $[A, A^*] = 0$

Compact if for any bounded sequence $\{x_n\}$ in $\mathcal{H}$ the sequence $\{Ax_n\}$ contains a convergent subsequence.
We also have that two operators A and B are said to λ commute if $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$.

We need the following properties of compact operators and some theorems for our results in this paper,

i. If A is compact then $A^*$is also compact.
ii. If A is compact then for any operator B we have that AB is also compact.
iii. If $\lambda \neq 0 \in \sigma(A)$ then $\lambda \in \sigma_p(A)$ thus in general we have that $\sigma(A) = \{0\}\, U\, \sigma_p(A)$.

Theorem *A* [1]
Let $A \in \mathcal{B}(\mathcal{H})$ be compact. If $0 \in W(A)$ then $W(A)$ is closed.

Theorem *B* [5]
If $A, B \in \mathcal{B}(\mathcal{H})$ are self adjoint λ-commuting operators with A compact, then we have that $W(AB)$ and $W(BA)$ are real

Theorem *C* [6]
Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators such that $0 \notin \overline{W(A)}$. Then we have $\sigma(A^{-1}B) \subset \overline{W(B)} - \overline{W(A)}$.

3. Main Results

Theorem 3.1
Let $A \in \mathcal{B}(\mathcal{H})$ be compact. Then $\sigma(A) \subset W(A)$ under any one of the following conditions
(i) $0 \notin \sigma(A)$
(ii) $0 \in W(A)$

Proof
We first note that in general $\sigma(A) \subset \overline{W(A)}$. However for A compact, $0 \notin \sigma(A)$ implies $\sigma_p(A) = \sigma(A)$. Thus every non-zero element of $\sigma(A)$ is an eigenvalue. But $\sigma_p(A) \subset W(A)$
(i) Indeed if $\lambda \neq 0 \in \sigma(A)$ .Then $Ax = \lambda x$ thus for x with $\|x\| = 1$ $\quad W(A) = \langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle = \lambda\|x\|^2 = \lambda$ i.e. $\lambda \in W(A)$
(ii) Now $0 \in W(A)$ implies $W(A)$ is closed by theorem A above. Thus $\overline{W(A)} = W(A)$ and hence $\sigma(A) \subseteq W(A)$.
The following is immediate

Corollary 3.2
Let $A \in \mathcal{B}(\mathcal{H})$ be a compact operator then $\sigma(A)$ and $W(A)$ are spectral sets under any of the conditions of **theorem 3.1** above

Corollary 3.3
Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that A is compact then we have:
(i) $0 \notin \sigma(AB)$ implies $\sigma(AB) \subset W(AB)$
(ii) $0 \in W(AB)$ implies $\sigma(AB) \subset W(AB)$

Proof
(i) We first note that A is compact implies, AB is compact now by part (i) of theorem 3.1 above $\sigma(AB) = \sigma_p(AB) \subset W(AB)$.
(ii)Similarly by part (ii) of theorem 3.1 above we have $0 \in W(AB)$ implies $W(AB)$ is closed. Hence $\sigma(AB) \subset W(AB) = \overline{W(AB)}$.

Corollary 3.4
Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that A is compact if $0 \notin \sigma(AB)$ and $0 \in W(BA)$ with either A or B invertible then we have that $\sigma(AB) = \sigma(BA) \subset W(AB)$ and $\sigma(AB) = \sigma(BA) \subset W(BA)$.

Proof
For definiteness we assume that A is invertible then we have $BA = A^{-1}(AB)A$ which implies AB and BA are similar. Hence $\sigma(AB) = \sigma(BA)$ .Thus by corollary 3.3 above we have $\sigma(AB) = \sigma(BA) \subset W(AB)$ and $\sigma(AB) = \sigma(BA) \subset W(BA)$.

Remark 3.5
We note that for an operator A we have that $W(A)$ is real does not necessarily imply $\sigma(A)$ is also real. However in view of rem B above we have the following result.

Theorem 3.6
Let $A, B \in \mathcal{B}(\mathcal{H})$ be self adjoint $\lambda-$ commuting operators with A compact if $0 \notin \sigma(AB)$ and $0 \in W(BA)$ then we have that $\sigma(AB)$ and $\sigma(BA)$ are real.

Proof
We first note that since A is compact both AB and BA are compact. By theorem C above A and B are $\lambda-$commuting are λ self adjoint operators implies $W(AB)$ and $W(BA)$ are real
Now $0 \notin \sigma(AB)$ implies $\sigma(AB) = \sigma_p(AB) \subset W(AB)$. Hence $\sigma(AB)$ is real.
Also $0 \in W(BA)$ implies $W(BA) = \overline{W(BA)}$
Thus $\sigma(BA) \subset W(BA)$. Hence $\sigma(BA)$ is also real.

Remark 3.7
We note that in theorem C above if the operator A is compact then we can relax the condition $0 \notin \overline{W(A)}$ and still same proof carries through as the result shows below.

Theorem 3.8
Let $A \in \mathcal{B}(\mathcal{H})$ be compact such that $0 \in W(A)$ and $0 \notin \sigma(A)$. Then for any other operator B we have $\sigma(A^{-1}B) \subset \overline{W(B)} - W(A)$.

Proof
We first note that $0 \notin \sigma(A)$ implies A^{-1} exists and the identity $\quad A^{-1}B - \lambda = A^{-1}(B - \lambda A)$ shows that if $\lambda \in \sigma(A^{-1}B)$ then $0 \in \sigma(B - \lambda A)$. Thus we have $0 \in \overline{W(B - \lambda A)} \subset \overline{W(B)} - \lambda W(A)$.
Thus $\lambda \in \overline{W(B)} - W(A)$ .Hence $\sigma(A^{-1}B) \subset \overline{W(B)} - W(A)$.

Corollary 3.9
If in **theorem 3.8** above the operator B is also compact with $0 \in W(B)$ then we have
$\sigma(A^{-1}B) \subset W(B) - W(A)$.

Proof
In this case $0 \in W(B)$ implies $W(B)$ is closed and hence $W(B) = \overline{W(B)}$ .Hence the results follows.
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