International Journal of Statistics and Applied Mathematics 2020; 5(4): 112-114

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2020; 5(4): 112-114

www.mathsjournal.com
Received: 17-05-2020
Accepted: 20-06-2020

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On unitary equivalence and almost similarity of some classes of operators in Hilbert spaces

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Abstract
In operator theory, it is a well-known fact that two similar operators have equal spectra even if they do not belong to the same class of operators. However, under a stronger relation of unitary equivalence it can be shown that two unitarily equivalent operators may belong to the same class of operators. In this paper our task is to show some results on such operators which belong to the same class and not only unitary equivalence but also under isometric equivalence. We also endeavor to extend this consideration to almost similarity of operators.

AMS Subject Classification: 47B47, 47A30, 47B20

Keywords: Unitary equivalence, Almost similarity and Posinormal operators.

1. Introduction
Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $X \in \mathcal{B}(\mathcal{H})$ is said to be a quasiaffinity if X is both one to one and has dense range. Also, two operators A and B are said to be similar if there exists an invertible operator S such that $A = S^{-1}BS$. If there exists a unitary operator U such that $A = U^*BU$, then A and B are said to be unitarily. On the other hand, A and B are said to be quasisimilar if there exists quasi affinities X and Y such that $AX = XB$ and $BY = YA$. Also, A and B are said to be almost similar, written as $A \overset{a.s}{\approx} B$ if there exists an invertible operator N such that
$A^*A = N^{-1}B^*BN$ and $A^* + A = N^{-1}(B^* + B)N$.
These properties of unitary equivalence, similarity, quasi similarity and almost similarity have been studied by various authors who by a large extent relate them to equality of spectra for such operators. Trivially similar operators or unitarily equivalent operators have equal spectra. However, the case of quasi similarity requires the two operators belong to a certain class of operators for them to have equal spectra.
W.S. Clary [2] proved that quasi similar hyponormal operators have equal spectra. R.G. Douglas [4] showed that quasi-similar normal operators are unitarily equivalent. B.M. Nzimbi *et al.* [9] showed that if a normal operator T is unitarily equivalent to another operator S, then S is also normal. Recently S.W. Luketero and J.M. Khalagai [8] considered the situation where two operators which are isometrically or coisometrically equivalent happen to belong to the same class. To this end, they were able to show that if an operator A is either isometrically or coisometrically equivalent to another operator B and A is binormal, then so is B.
In this short communication we continue with this consideration and also endeavor to extend it to almost similarity relation. More specifically we consider the classes of θ-operators and posinormal operators. Also see [7].

2. Notations, Definitions and Terminologies
Given a complex Hilbert space $\mathcal{H}$ and operators $A, B \in \mathcal{B}(\mathcal{H})$ the commutator of A and B is denoted by $[A, B] = AB - BA$. Thus $[A, B] = 0$ implies A and B commute. The range and kernel of A are denoted by Ran A and Ker A respectively. The spectrum of A is denoted by $\sigma(A) = \{\lambda \in \mathbb{C}: A - \lambda I \text{ is not invertible}\}$.

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be:

- Normal if$[A, A^*] = 0$
- Quasinormal if$[A^*, A] = 0$
- Hyponormal if$A^*A \geq AA^*$
- Binormal if$[A^*A, AA^*] = 0$
- Posinormal if$AA^* = A^*PA$ for $P \geq 0$
- Partial isometry if $A = AA^*A$
- Isometry if$A^*A = I$
- Coisometry if $AA^* = I$
- Unitary if $A^*A = AA^* = I$
- θ-operator or $A \in \theta$ if $[A^*A, A^*{+}A] = 0$
- Dominant if for each $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$(A-\lambda)(A-\lambda)^* \leq M_\lambda (A-\lambda)^*(A-\lambda)$. If the constants M_λ are bounded by a positive number M, then A is said to be M-hyponormal.

We also have the following inclusions of classes of operators;

{Normal}⊆{Quasinormal}⊆ {Binormal}

{Normal}⊆ θ

{Unitary}⊆{Isometry}⊆{Partial isometry}

{Unitary}⊆{Coisometry}⊆ {Partial isometry}

{Normal}⊆{Hyponormal}⊆{M-hyponormal}⊆ {Dominant}⊆{Posinormal}

{Set of invertible operators}⊆{Set of posinormal operators}

3. Main results

3.1 Theorem

Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that B is a θ-operator and $A = UBU^*$where U is an isometry. Then A is also a θ-operator.

Proof

Since B is a θ-operator we have

$B^*B(B^* + B) = (B^* + B)B^*B$ i.e

$B^*BB^* + B^*B^2 = B^{*2}B + BB^*B ----(1)$.

But $A = UBU^*$ implies $A^* = UB^*U^*$

Therefore, $A^*A = UB^*U^*UBU^* = UB^*BU^*$ and $A^* + A = UB^*U^* + UBU^*$. Thus we have,

$A^*A(A^* + A) = UB^*BU^*(UB^*U^* + UBU^*)$

$= UB^*BB^*U^* + UB^*B^2U^* ----(2)$.

Also, $(A^* + A)A^*A = (UB^*U^* + UBU^*)UB^*BU^*$

$= UB^{*2}BU^* + UBB^*BU^* ----(3)$.

Using (1) we have that, $UB^*BB^*U^* + UB^*B^2U^* = UB^{*2}BU^* + UBB^*BU^*$.

From (2) and (3) we have $A^*A(A^* + A) = (A^* + A)A^*A$. Hence, $A$is also a θ-operator.

3.2 Theorem

Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that B is a θ-operator and $A = U^*BU$ where U is a coisometry. Then A is also a θ-operator.

Proof

Since B is a θ-operator we have that;

$B(B^* + B) = (B^* + B)B^*B$

$B^*BB^* + B^*B^2 = B^{*2}B + BB^*B ----(1)$.

But $A = U^*BU$ implies $A^* = U^*B^*U$. Therefore,

$A^*A = U^*B^*UU^*BU = U^*B^*BU$ and

$A^* + A = U^*B^*U + U^*BU$.

Thus we have,$A^*A(A^* + A) = U^*B^*BU(U^*B^*U + U^*BU)$

$= U^*B^*BB^*U + U^*B^*B^2U ----(2)$ and

$(A^* + A)A^*A = (U^*B^*U + U^*BU)U^*B^*BU$

$= U^*B^{*2}BU + U^*\,B\,B^*BU ----(3)$,

From (1)we have that;

$U^*\,B^*B\,B^*U + U^*\,B^*B^{*2}U = U^*B^{*2}BU + U^*BB^*BU$. Hence from (2) and (3) we have;

$A^*A(A^* + A) = (A^* + A)A^*A$. Thus $A \in \theta$.

3.3 Corollary

Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that B is a θ-operator and either $A = U^*BU$ or $A = UBU^*$, where U is unitary .Then A is also a θ-operator.Thus A and B are unitarily equivalent θ-operators.

Proof

From the inclusions of classes of operators we note that every unitary operator is either an isometry or coisometry. Hence by both theorems 3.1 and 3.2 above the result follows easily.

3.4 Remark

We note that we have similar situation when we replace the class of θ-operators in the theorems above with the class of posinormal operators as it is shown in the results below.

3.5 Theorem

Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that A is posinormal and either

(i) $A = UBU^*$ with U isometryor

(ii) $A = U^*BU$ with U coisometry.

Then B is also posinormal.

Proof

(i) Since A is posinormal we have $AA^* = A^*PA$ for $P \geq 0$.

But $A = UBU^*$ implies

$A^* = UB^*U^*$.Therefore, $AA^* = UBU^*UB^*U^* = UBB^*U^*$and

$A^*PA = UB^*U^*PUBU^* = UB^*(U^*PU)BU^*$.

Thus, $UBB^*U^* = UB^*(U^*PU)BU^* ----(1)$.

Now premultiplying by U^* and post multiplying by U in (1) gives

$BB^* = B^*(U^*PU)B$ where, $U^*PU \geq 0$. Hence B is also posinormal.

(ii) Since A is posinormal we have that, $AA^* = A^*PA$. But $A = U^*BU$ implies $A^* = U^*B^*U$.

Therefore, $AA^* = U^*BUU^*B^*U = U^*BB^*U$and

$A^*PA = U^*B^*UPU^*BU = U^*B^*(UPU^*)BU$.Thus,

$U^*BB^*U = U^*B^*(UPU^*)BU ----(1)$

Now premultiplying by U and postmultiplying by U^* in (1) gives

$BB^* = B^*(UPU^*)B$, where $UPU^* \geq 0$. Hence, B is also posinormal.

3.6 Corollary

Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that A is posinormal and either $A = UBU^*$ or $A = U^*BU$ where U is unitary. Then B is also posinormal. Thus A and B are unitarily equivalent posinormal operators.

Proof

Since every unitary operator is either an isometry or coisometry, the result follows immediately from theorem 3.5 above.

3.7 Remark

We note from the results above that the classes of θ-operators and posinormal operators are not only unitarily invariant but also isometrically and co isometrically invariant.

We also note that with respect to almost similarity of operators if two operators are almost similar with one of them a θ-operator then the other one is also a θ-operators, see [1].

The same statement cannot be made about posinormal operators. However, we have the following result.

3.8 Theorem

Let $A, B \in \mathcal{B}(\mathcal{H})$ be almost similar operators with their polar decompositions as $A = U|A|$ and $B = V|B|$ where U and V are unitary. Then A is invertible implies B is also invertible.

Proof

Since $A \overset{a.s}{\approx} B$, there exists an invertible operator N such that $A^*A = N^{-1} B^*BN$ and

$A^* + A = N^{-1} (B^* + B)N$. Now A is invertible implies A^* is also invertible which also implies A^*A is invertible. But A^*A is similar to B^*B which implies that $\sigma(A^*A) = \sigma(B^*B)$. Thus B^*B is an invertible positive operator which implies that $\sqrt{B^*B} = |B|$ is also invertible. Hence B is invertible.

3.9 Remark

We note that the class of invertible operators is a subclass of posinormal operators, see [3].

We need the following definition for our next result.

3.10 Definition

For an operator $B \in \mathcal{B}(\mathcal{H})$, we say that B is *consistent in invertibility* (with respect to multiplication) or briefly that B is CI operator if for any other operator

$A \in \mathcal{B}(\mathcal{H})$, we have that AB and BA are invertible or non-invertible together. Thus B is a CI operator if $\sigma(AB) = \sigma(BA)$.

It is a well-known result that if B is invertible, then for any operator $A \in \mathcal{B}(\mathcal{H})$, we have

$AB = B^{-1} (BA)B$. Thus AB and BA are similar and hence $\sigma(AB) = \sigma(BA)$. To this end,

W. Gong and D. Han [5] showed that an operator $B \in \mathcal{B}(\mathcal{H})$ is CI if and only if

$\sigma(B^*B) = \sigma(BB^*)$.

3.11 Theorem

Let $A \in \mathcal{B}(\mathcal{H})$ be almost similar to A^*. Then A is a consistent invertible operator.

Proof

Since $A \overset{a.s}{\approx} A^*$, we have $A^*A = N^{-1}A\,A^*N$ for some invertible operator N. Thus A^*A is similar to $A\,A^*$ which implies $\sigma(A^*A) = \sigma(AA^*)$. Hence A is a consistent invertible operator.

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