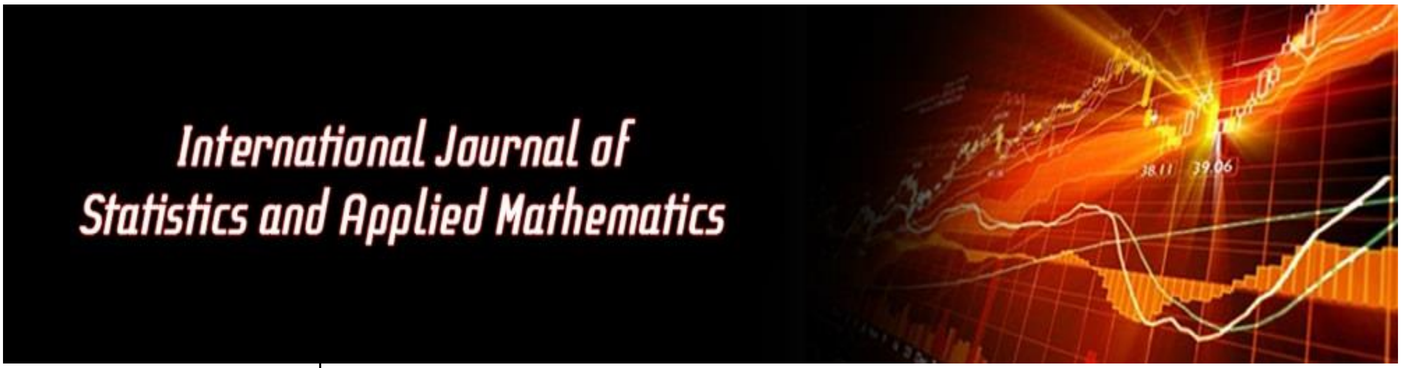

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Invariant subspaces of bilateral shift on $L^2(S^1)$

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Abstract
In this Paper, we have proved that all the invariant subspaces of the bilateral shift on $L^2(S^1)$ is of the form φH_b^2 where φ is a function in L^∞ which is equals to 1 almost everywhere.

Keywords: Measurable, essentially bounded, commutant, unitary

1. Introduction

Notation $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ i.e S^1 is the circle with center origin and radius 1.

1.1 Definition ($L^2(S^1)$)

It is defined as the space of all the equivalence classes of functions [3] that are Lebesgue measurable on S^1 and square integrable on S^1 with respect to Lebesgue measure normalized such that measure of S^1 is 1.

$$L^2(S^1) = \{f : f \ \ is \ \ Lesbesgue \ \ measurable \ \ on \ \ \mathbb{S}^{\Bbbk} \ \ and \ \ \frac{1}{2\pi}\int_0^{2\pi} |f(e^{\iota\theta})|^2 d\theta < \infty\}$$

Inner product on $L^2(S^1)$ is given by –

$$< f, g >= \frac{1}{2\pi}\int_0^{2\pi} f(e^{\iota\theta})\overline{g(e^{\iota\theta})}d\theta$$

Note $L^2(S^1)$ is an Hilbert-space with the orthonormal basis given by $\{e_n : n \in \mathbb{Z}\}$ where $e_n(e^{\iota\theta}) = e^{\iota n\theta}$.

Therefore

$$L^2(S^1) = \left\{f : f = \sum_{n=-\infty}^{n=\infty} < f, e_n > e_n\right\}.$$

1.2 Definition (H_c^2 space)

$H_c^2 = \{f \in L^2(S^1) :< f, e_n >= 0 \ for \ negative \ value \ of \ n\}$

(∞)

$H_c^2 = f \in L^2(S^1) : f = {}^{X} < f, e_n > e_n. \ n=0$

$\widehat{H^2}$ is a closed subspace of $L^2(S^1)$ whose negative Fourier coefficients are 0

$\therefore \{e_n : n = 0,1,...\}$ are orthonormal basis of $\widehat{H^2}$

1.3 Wandering Subspace

A closed N of a Hilbert-Space H is said to be a Wandering subspace of a bounded linear operator T on H if N is orthogonal to $T^k(N)$ for all $k \geq 0$

1.4 The Wolde's Decomposition Theorem

Let T be an isometric operator on a Hilbert-Space H then

$$H = \bigcap_{k=0}^{\infty} T^k(H) \bigoplus N \bigoplus T(N) \bigoplus \ldots\ldots$$

Where
$H = T(H)^M N$

1.4 Definition (L^∞)

It is defined as the Collection of all the essentially bounded measurable functions on the circle S^1.
For a function $\varphi \in L^\infty$ the essential norm is defined by

$||\varphi||_\infty = inf\{r : m\{e^{\iota\theta}; |\varphi(e^{\iota\theta})| > r\} = 0\}$

where m is normalized lesbesgue measure.

Note

$|\varphi(e^{\iota\theta}| \leq ||\varphi||_\infty \, a.e$

1.6 Multiplication Operator by a L^∞ function

For a function $\varphi \in L^\infty$ the multiplication operator on $L^2(S^1)$ is defined by $S_\varphi : L^2(S^1) \rightarrow L^{(}S^1)$

$f \rightarrow \varphi f$

Note

$$\int_0^{2\pi} |\phi(e^{\iota\theta})|^2 |f(e^{\iota\theta})|^2 d\theta \leq ||f||^2 ||\phi||_\infty^2 < \infty \;\; since \;\; (|\phi(e^{\iota\theta})| \leq ||\phi||_\infty a.e)$$

$\Rightarrow \varphi f \in L^2(S^1)$

$\therefore S_\varphi$ is well defined

1.7 Bilateral Shift

Bilateral shift on $L^2(S^1)$ is an operator $B : L^2(S^1) \rightarrow L^2(S^1)$ defined by

$Bf(e\iota\theta) = e\iota\theta f(e\iota\theta)$

Note B is an isometric Unitary Operator [2] whose adjoint is the bounded linear operator $B^? : L^2 \rightarrow L^2$ given by

$B?f(e\iota\theta) = e{-}\iota\theta f(e\iota\theta)$
i.e. $B^?B = I = BB^?$

1.8 Invariant Subspace

A Closed subspace N of a Hilbert-Space H is said to be an Invariant Subspace of s Bounded linear operator T on H if $T(N) \subseteq N$

Theorem 1.1. *N is an invariant subspace of T iff* $N^\perp$ *is an invariant subspace of* $T^?$
Proof. Let N is an invariant subspace of T

$\Rightarrow T(N) \subseteq N$

Let $y \in T^?(N^\perp)$ be arbitrary then $\exists\, x \in N^\perp$ such that

$y = T^?(x)$

$\Rightarrow \;\; < y,n > = < T^?(x),n > = < x,T(n) > = \;\; 0 \quad \forall\, n \in N \qquad (\because T(N) \subseteq N)$
$\Rightarrow y \in N^\perp \Rightarrow \;\; T^?(N^\perp) \subseteq N^\perp$
$\therefore N^\perp$ is an invariant subspace of $T^?$
Conversely

Let $N^\perp$ is an invariant subspace of $T^?$
$\Rightarrow T^?(N^\perp) \subseteq N^\perp$

Let $y \in T(N)$ be arbitrary then $\exists\, x \in N$ such that
$\Rightarrow y = T(x)$
$\Rightarrow < y,n > = < T(x),n > = < x,T^{?}(n) > = 0\ \forall\ n \in N^{\perp}$ $\qquad (\because T^{?}(N^{\perp}) \subseteq N^{\perp})$
$y \in N \Rightarrow T(N) \subseteq N$
∴ N is an invariant subspace of T

1.9 Reducing Subspace
An invariant Subspace N of a bounded linear operator T on a Hilbert Space H if

$T(N) \subseteq N$ *and* $T(N^{\perp}) \subseteq N^{\perp}$

i.e. $T(N) \subseteq N$ *and* $T^{?}(N) \subseteq N$

Theorem 1.2. *If N is a reducing subspace of a unitary operator T on a hilbert space H then* $T(N) = N$
Proof. Since N is a reducing subspace of T

$\Rightarrow \quad T(N) \subseteq N$ *and* $T^{?}(N) \subseteq N$ [1]

Since T is a Unitary Operator

$\Rightarrow TT^{?} = I = T^{?}T$

Then
$N = TT^{?}(N) \subseteq T(N)\ (\because T^{?}(N) \subseteq N)$
$\Rightarrow T(N) = N$

Result (?) If E is a non zero Reducing Subspace of bilateral shift B on $L^2(S^1)$ then $\exists$ a subset E of S^1 of positive measure such that

$E = X_E L^2(S^1)$

2. Invariant subspace of bilateral shift on $L^2(S^1)$
Theorem 2.1. *Let N be an invariant(not reducing) subspace of bilateral shift B on* $L^2(S^1)$ *then there exist a function φ in* L^{∞} *such that* $N = \varphi H_c^2$ *with essential norm 1.*
Proof. Since N is an Invariant but not reducing subspace of bilateral shift $B : L^2(S^1) \rightarrow L^2(S^1)$

$\Rightarrow B(N)\ (\ N \quad (\because Theorem 1.2)$

Since B is an Isometry and N is a closed subspace of $L^2(S^1)$. Then by Wolde's Decomposition Theorem
(1.3) we have

$$N = \bigcap_{n=0}^{\infty} B^k(N)^{M} K\ ^{M}B(K)^{M}B^2(K)^{M}.........$$

Where $N = K\ ^{L}B(N)$ ($\because$ since B is an isometry then B(N) is a closed) Since K 6= {0} then we can find $\varphi \in N$ such that $||\varphi||_L 2 = 1$

1. Claim : φ is a function in L^{∞} with essential norm 1
Since $\varphi \in K \subset B(N)^{\perp}$

$$\Rightarrow \phi \perp B^k(\phi) \quad for \quad all \quad k \in \mathbb{N} \qquad (\because B^k(N) \subseteq B(N))$$

$$\Rightarrow < \phi, B^k(\phi) >= 0 \qquad \forall \quad k \in \mathbb{N}$$

$$\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{\iota\theta})|^2 e^{k\iota\theta} d\theta = 0 \ \ \forall \ \ k \in \mathbb{N} \qquad (1) \tag{1}$$

Taking conjugate of (1) we get

$$\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{\iota\theta})|^2 e^{-k\iota\theta} d\theta = 0 \ \ \forall \ \ k \in \mathbb{N} \tag{2}$$

Since $\varphi \in L^2 \Rightarrow |\varphi|^2 \in L^1(S^1)$ Then from(1) and (2) we have

$< |\varphi|^2, e_k >= 0\ \forall\ k = \pm 1, \pm 2, ...$

And

$$||\phi||_{L^2} = 1 \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{\iota\theta})|^2 d\theta = 1 \Rightarrow < |\phi|^2, e_0 >= 1$$
$$\Rightarrow |\phi(e^{\iota\theta}| = 1 \ \ a.e.$$

$\therefore \varphi$ is a function in L^∞ with essential norm 1.

2. Claim: K = span(φ)= < φ >

Suppose not then $\exists$ a function $\psi \in K$ such that $\psi \perp \varphi$ and $||\psi||_L 2 = 1$
Then same as claim 1 we can also show that ψ is a function in L^∞ with essential norm 1 Since

$\varphi \perp \psi \Rightarrow \varphi \perp B^k(\psi)$ *and* $B^k(\varphi) \perp \psi \ \forall \ k \in \mathrm{N}$ $(\because B^K(N) \subset B(N))$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{\iota\theta})\overline{\psi(e^{\iota\theta})}e^{-k\iota\theta}d\theta = 0 \ \ \forall \ \ k = 0, \pm1, \pm2, ...$$
$$\Rightarrow \phi\psi = 0$$

which is a contradiction since $|\varphi(e^{\iota\theta})| = 1 = |\psi(e^{\iota\theta}|$ *a.e.*

$\therefore K = span(\varphi) =< \varphi >$

Then we have

$$N = \bigcap_{n=0}^{\infty} B^k(N) \bigoplus < \phi > \bigoplus B(< \phi >) \bigoplus B^2(< \phi >) \bigoplus ....$$

3. Claim: $< \phi > \bigoplus B(< \phi >) \bigoplus B^2(< \phi >) \bigoplus .... = \phi\widehat{H^2}$

Let let $f \in L.H.S$ then
$f = \alpha_1\varphi + \alpha_2\varphi e_1 + \alpha_3\varphi e_2 + ... \in \varphi H_c^2$ (∵ *By Definition*1.2)
Conversely , let $f \in \varphi H_c^2$
∞

$\Rightarrow f =$ X $< f,e_n > \varphi e_n$

n=0

which clearly lies in the L.H.S. Hence the claim Then

∞
N = \ $Bk(N)$MφHc2
n=0

4. Claim: $N = \phi\widehat{H^2}$
Clearly
∞
\ k
B (N)
n=0

is a reducing subspace of bilateral shift B. Then by the Result (?) $\exists$ an subset E of S^1 such thatt

$$\bigcap_{n=0}^{\infty} B^k(N) = X_E L^2$$

$$\Rightarrow N = X_E L^2 \bigoplus \phi\widehat{H^2}$$

$$\Rightarrow X_E\phi \perp \phi \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} X_E(e^{\iota\theta})|\phi(e^{\iota\theta})|^2 d\theta = 0$$

$$\Rightarrow \frac{1}{2\pi}\int_E |\phi(e^{\iota\theta})|^2 d\theta = 0$$

⇒ $|\varphi|$ is zero on E

But
$|\varphi| = 1 \quad \Rightarrow m(E) = 0$ *a.e.*

Hence
$N = \phi\widehat{H^2}$

where φ is a function in L^∞ of essential norm 1.

3. Conclusion

We proved the results that help to identify all the invariant subspaces of Bilateral shift on L^2 which is of the form φH_b^2 where φ is a function in L^∞ which is equals to 1 almost everywhere which results to make the study of theseinvariant subspaces much easier.

4. References

1. Erwin Kreyszig. Introductory functional analysis with applications, wiley New York, 1978, 1.
2. Rub´en Mart´ınez-Avendan˜o A, Peter Rosenthal. An introduction to operators on the Hardy-Hilbert space, 237. Springer Science & Business Media, 2007.
3. Halsey Lawrence Royden et al. Real analysis. Prentice Hall, 2010.