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Solve two kinds of generalized integrals by using gamma function

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Abstract
This paper uses the calculation formula of the gamma function to cleverly deal with two types of generalized integrals, and further generalizes, and gives more general solution ideas and conclusions.

Keywords: Gamma function normal distribution improper integrals gauss integration

Introduction

When learning the course of probability theory, it is often necessary to prove that the generalized integral containing e^{-x^2} is often required in the process of proof. Since this form of function does not have a primitive function that can be expressed by elementary functions, it cannot be solved directly by the Newton-Leibniz formula.

The general solution idea is to use the double integral to solve the expression square or use the standard normal distribution integral to replace the solution, because the normal distribution density function integral is equal to 1.

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1 \tag{1}$$

Generally, (1) can be obtained by simply changing yuan, let $\frac{x-\mu}{\sigma} = t$. We can get the following formula

$$\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi} \tag{2}$$

However, the limitation of using normal distribution law is that the coefficient must be $-\frac{1}{2}$, and then combined with the parity of the product function, the calculation formula (3) of the gamma function can further popularize the coefficient of t^2, and can simplify the thinking and steps of calculation. Solve the following two types of anomalous integrals quickly and are easy to accept.

$$\int_{0}^{+\infty} x^{n-1} e^{-x} dx = \Gamma(n) \tag{3}$$

2. Two types of generalized integrals.
2.1 The first category of generalized integral
(1) Sample question

$$\int_{-\infty}^{+\infty} x^k e^{-\lambda x^2} dx$$

Solution

1. When k is odd, because the product function is an odd function, the integral interval is about the origin symmetry, by the nature of the integral can be known.

$$\int_{-\infty}^{+\infty} x^k e^{-\lambda x^2} dx = 0$$

2. When k is even, because the product function is an even function, the integral interval is symmetrical about the origin, which is available by the integral nature.

$$\int_{-\infty}^{+\infty} x^k e^{-\lambda x^2} dx = 2\int_0^{+\infty} x^k e^{-\lambda x^2} dx$$

So, we just need to solve the value of $\int_0^{+\infty} x^k e^{-\lambda x^2}\, dx$

When k is equal to zero, $\int_0^{+\infty} e^{-\lambda x^2} dx = \frac{1}{2\lambda}\int_0^{+\infty} x^{-1} e^{-\lambda x^2} d(\lambda x^2)$, let ? $x^2 = t$,

$$\int_0^{+\infty} \frac{1}{2\lambda}\int_0^{+\infty} x^{-1} e^{-\lambda x^2} d(\lambda x^2) = \frac{1}{2\sqrt{\lambda}}\int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt = \frac{1}{2\sqrt{\lambda}}\Gamma(\frac{1}{2})$$

When k is equal to one, $\int_0^{+\infty} x e^{-\lambda x^2} dx = \frac{1}{2\lambda}\int_0^{+\infty} e^{-\lambda x^2} d(\lambda x^2)$ let ? $x^2 = t$,

$$\frac{1}{2\lambda}\int_0^{+\infty} e^{-\lambda x^2} d(\lambda x^2) = \frac{1}{2\lambda}\int_0^{+\infty} t^0 e^{-t} d(t) = \frac{1}{2\lambda^{\frac{2}{2}}}\Gamma(\frac{2}{2})$$

When k is equal to two, $\int_0^{+\infty} x^2 e^{-\lambda x^2} dx = \frac{1}{2\lambda}\int_0^{+\infty} x e^{-\lambda x^2} d(\lambda x^2)$, let ? $x^2 = t$,

$$\int_0^{+\infty} x^2 e^{-\lambda x^2} dx = \frac{1}{2\lambda}\int_0^{+\infty} x e^{-\lambda x^2} d(\lambda x^2) = \frac{1}{2\sqrt{\lambda}}\int_0^{+\infty} t^{\frac{1}{2}} e^{-t} d(t) = \frac{1}{2\lambda^{\frac{3}{2}}}\Gamma(\frac{3}{2})$$

Suppose when k is equal to $n-1$, $\int_0^{+8} x^{n-1} e^{-?x^2} dx = \frac{1}{2?^{\frac{n}{2}}} G\left(\frac{n}{2}\right)$, then when k is equal to n, $\int_0^{+8} x^n e^{-?x^2} dx = \int_0^{+8} x^{m-1} e^{-?x^2} dx = \frac{1}{2?^{\frac{m}{2}}} G\left(\frac{m}{2}\right) = \frac{1}{2?^{\frac{n+1}{2}}} G\left(\frac{n+1}{2}\right)$.

This formula also conforms to the form. From mathematical induction, we can know

$$\int_0^{+8} x^{n-1} e^{-?x^2} dx = \frac{1}{2?^{\frac{n}{2}}} G\left(\frac{n}{2}\right) \tag{4}$$

So, when k is even, we can get the following formula,

$$\int_{-8}^{+8} x^k e^{-?x^2} dx = 2\int_0^{+8} x^k e^{-?x^2} dx = \frac{1}{?^{\frac{k+1}{2}}} G\left(\frac{k+1}{2}\right)$$

In summary,

$$\int_{-\infty}^{+\infty} x^k e^{-\lambda x^2} dx = \begin{cases} \frac{1}{\lambda^{\frac{k+1}{2}}} \Gamma\left(\frac{k+1}{2}\right) & k = 2n \\ & k,n \in z \\ 0 \quad k = 2n+1 \end{cases}$$

(2) Further promotion

$$\int_{-\infty}^{+\infty} (a^x)^k e^{-\lambda (a^x)^2} dx$$

(i) If a is greater than one, let $a^x = t$, $dx = \frac{1}{t \ln a} dt$

then $\int_{-\infty}^{+\infty} (a^x)^k e^{-\lambda (a^x)^2} dx = \frac{1}{\ln a} \int_{0}^{+\infty} (t)^{k-1} e^{-\lambda (t)^2} dt$

From the conclusion in (4), we can know

$$\int_{-\infty}^{+\infty} (a^x)^k e^{-\lambda (a^x)^2} dx = \frac{1}{\ln a} \frac{1}{2\lambda^{\frac{k}{2}}} \Gamma(\frac{k}{2}) \quad (5)$$

(ii) If a is greater than zero and less than one, let $a^x = t$, $dx = \frac{-1}{t \ln a} dt$

Then

$$\int_{-\infty}^{+\infty} (a^x)^k e^{-\lambda (a^x)^2} dx = \frac{-1}{\ln a} \int_{+\infty}^{0} (t)^{k-1} e^{-\lambda (t)^2} dt = \frac{1}{\ln a} \int_{0}^{+\infty} (t)^{k-1} e^{-\lambda (t)^2} dt$$

From the conclusion in (4), we can know

$$\int_{-\infty}^{+\infty} (a^x)^k e^{-\lambda (a^x)^2} dx = \frac{1}{\ln a} \frac{1}{2\lambda^{\frac{k}{2}}} \Gamma(\frac{k}{2}) \quad (6)$$

(3) Further promotion

$$\int_{-\infty}^{+\infty} (\frac{a}{x})^k e^{-\lambda (\frac{a}{x})^2} dx$$

let $\frac{a}{x} = t$, $\int_{-\infty}^{+\infty} (\frac{a}{x})^k e^{-\lambda (\frac{a}{x})^2} dx = -a \int_{-\infty}^{+\infty} t^{k-2} e^{-\lambda t^2} dt$

Using the conclusion in (1) directly, we can know

$$-a \int_{-\infty}^{+\infty} t^{k-2} e^{-\lambda t^2} dt = \begin{cases} -\frac{a}{\lambda^{\frac{k+1}{2}}} \Gamma\left(\frac{k+1}{2}\right) \quad k = 2n \\ \\ 0 \quad k = 2n+1 \quad k,n \in z \end{cases}$$

2.2 Generalized integral of the second kind (integral of Gaussian function)
As we all know, Gaussian function belongs to elementary function, but it has no elementary indefinite integral. But it is still possible to calculate its generalized integral on the entire real number axis.

(1) Standard form of Gaussian integral

$$\int_{-\infty}^{+\infty} A\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

Among them, the parameter A refers to the peak value of the Gaussian curve, μ is the corresponding abscissa, and σ is the standard deviation. The conclusion of the standard normal distribution can be borrowed, namely, $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$ = 1 ?

$$\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\sqrt{2ps} \qquad (9)$$

So, $\int_{-\infty}^{+\infty} A\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = A\cdot\sqrt{2\pi}\sigma$

(2) General expression of Gauss integral

$$\int_{-\infty}^{+\infty} Ag e^{ax^2+bx+c}dx(a<0)$$

Solution
We can get the following formula by matching method,

$$\int_{-\infty}^{+\infty} A\cdot e^{ax^2+bx+c}\,dx=\int_{-\infty}^{+\infty} A\cdot e^{-[(\sqrt{-a}x-\frac{b}{2\sqrt{-a}})^2-(c-\frac{b^2}{4a})]}\,dx$$

let $\sqrt{-a}x-\frac{b}{2\sqrt{-a}}=t$

$$\int_{-\infty}^{+\infty} A\cdot e^{ax^2+bx+c}dx=\frac{Ae^{c-\frac{b^2}{4a}}}{\sqrt{-a}}\int_{-8}^{+8}e^{-t^2}dt = Ae^{c-\frac{b^2}{4a}}\sqrt{\frac{p}{-a}}$$

(3) Further extend the coefficient A to the nth-order polynomial form to solve the integral

$$\int_{-\infty}^{+\infty} P(x)g e^{ax^2+bx+c}dx \qquad P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x^1+a_0.$$

Solution

$$\int_{-\infty}^{+\infty} P(x)g e^{ax^2+bx+c}dx=\int_{-\infty}^{+\infty}(a_0x^n+a_1x^{n-1}+\mathrm{L}\ +a_{n-1}x^1+a_0)g e^{ax^2+bx+c}dx$$

$$=\int_{-\infty}^{+\infty}a_0x^n g e^{ax^2+bx+c}dx+\int_{-\infty}^{+\infty}a_1x^{n-1}g e^{ax^2+bx+c}dx+\mathrm{L}\ +\int_{-\infty}^{+\infty}a_{n-1}x^1 g e^{ax^2+bx+c}dx+\int_{-\infty}^{+\infty}a_n g e^{ax^2+bx+c}dx$$

In short, after formulating the index of e for each term in the formula, repeat the step of substituting in **(2)** in **2.1**, and then use the calculation formula of the first type of generalized integral summary to obtain the result.

3. In conclusion
This paper uses the gamma function to solve the two types of generalized integrals and summarizes the calculation formulas. These results are a series of popularization and application of formula (4), which are not only easy to understand but also quick to get results.

4. References
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