Measure of slope rotatability for second order response surface designs using a pair of symmetrical unequal block arrangements with two unequal block sizes

P Chiranjeevi and BRE Victorbabu

Abstract
A measure enables to assess the degree of slope rotatability for a given response surface design. In this paper, measure of slope rotatability for second order response surface designs using a pair of symmetrical unequal block arrangements with two unequal block sizes is recommended.

Keywords: Response surface designs, measure of slope rotatability for second order response surface designs, symmetrical unequal block arrangements with two unequal block sizes

1. Introduction

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BIBD, PBD, SUBA with two unequal block sizes and partially balanced incomplete block type designs, pair of BIBD respectively. Victorbabu and Chiranjeevi (2018) [24] studied measure of rotatability for second order response surface designs using SUBA with two unequal block sizes when \( r > 3\lambda \) is suggested. Chiranjeevi and Victorbabu (2019) [2] suggested measure of slope rotatability for second order response surface designs using supplementary difference sets. In this paper, a new measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes is suggested.

2. Conditions for second order slope rotatable designs

Suppose we want to use the second order response surface designs \( D = (x_{iu}) \) to fit the surface,

\[
Y_{u} = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j} b_{ij} x_{iu} x_{ju} + e_u
\]  

(1)

where \( x_{iu} \) denotes the level of the \( i \)th factor \((i=1,2,...,v)\) in the \( u \)th run \((u=1,2,...,N)\) of the experiment and the \( e_u \)'s are uncorrelated random errors with mean zero and variance \( \sigma^2 \). A second order response surface design \( D \) is said to be SOSRD if the variance of estimated of first order partial derivative of \( Y_u(x_1, x_2, ..., x_v) \) with respect to each of independent variable \((x_i)\) is only a function of the distance \( d^2 = \sum x_i^2 \) of the point \((x_1, x_2, ..., x_v)\) from the origin (center) of the design. Such a spherical variance function for estimation of slopes in the second order response surface is achieved. If the design points satisfy the following conditions. (cf. Hader and Park (1978) [4], Victorbabu and Narasimham (1991a) [13]).

\[
\begin{align*}
\sum x_{iu} &= 0, \sum x_{iu} x_{ju} = 0, \sum x_{iu} x_{ju}^2 = 0, \sum x_{iu} x_{ju} x_{ku} = 0, \\
\sum x_{iu}^3 &= 0, \sum x_{iu} x_{ju}^3 = 0, \sum x_{iu} x_{ju} x_{ku}^2 = 0, \sum x_{iu} x_{ju} x_{ku} x_{lu} = 0.
\end{align*}
\]

(2)

\( i \neq j \neq k \neq l \);

(i) \( \sum x_{iu}^2 = \text{constant} = N\lambda_2; \) (ii) \( \sum x_{iu}^4 = \text{constant} = cN\lambda_4; \) for all \( i \)

\( \sum x_{iu}^3 x_{ju} = \text{constant} = N\lambda_4; \) for all \( i \neq j \)

\( \frac{\lambda_4}{\lambda_2^2} > \frac{v}{(c+v-1)} \)  

(3)

\( 4 \lambda_4 \left[ v(c-3)^2 \right] + \lambda_2^2 \left[ v(c-5)+4 \right] = 0 \)

(4)

(5)

(6)

where \( c, \lambda_2 \) and \( \lambda_4 \) are constants and the summation is over the design points.

The variances and co-variances of the estimated parameters are,

\[
\begin{align*}
\hat{V}(\hat{b}_0) &= \frac{\lambda_4 (c+v-1)\sigma^2}{N[\lambda_4 (c+v-1)-v\lambda_2^2]}, \\
\hat{V}(\hat{b}_i) &= \frac{\sigma^2}{N\lambda_2}, \\
\hat{V}(\hat{b}_{ii}) &= \frac{\sigma^2}{N\lambda_4}, \\
\hat{V}(\hat{b}_{ij}) &= \frac{\sigma^2}{(c-1)N\lambda_4} \left[ \frac{\lambda_4 (c+v-2)-(v-1)\lambda_2^2}{\lambda_4 (c+v-1)-v\lambda_2^2} \right], \\
\hat{V}(\hat{b}_{ii}) &= \frac{\lambda_4 (c+v-2)-(v-1)\lambda_2^2}{\lambda_4 (c+v-1)-v\lambda_2^2}, \\
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\hat{V}(\hat{b}_{ii}) &= \frac{-\lambda_2 \sigma^2}{N[\lambda_4 (c+v-1)-v\lambda_2^2]}, \\
\hat{Cov}(\hat{b}_0, \hat{b}_i) &= \frac{-\lambda_{ij} \sigma^2}{N[\lambda_4 (c+v-1)-v\lambda_2^2]}, \\
\hat{Cov}(\hat{b}_i, \hat{b}_{ii}) &= \frac{-\lambda_{ij} \sigma^2}{N[\lambda_4 (c+v-1)-v\lambda_2^2]}, \\
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\hat{Cov}(\hat{b}_{ii}, \hat{b}_{ii}) &= \frac{-\lambda_{ij} \sigma^2}{N[\lambda_4 (c+v-1)-v\lambda_2^2]}.
\end{align*}
\]
\[ \text{Cov}(\hat{b}_i, \hat{b}_j) = \frac{(\lambda_i^2 - \lambda_j^2)\sigma^2}{(c-1)N\lambda_i(c+v-1)-v\lambda_i^2} \] and other covariances vanish. \hfill (7)

Therefore the conditions (2) to (7) give a set of conditions for slope rotatability in any general second order response surface design.

3. Second order slope rotatable designs using a pair of SUBA with two unequal block sizes (cf. Victorbabu and Narayanarao (2006) [10, 17])

Result: The method of construction of SOSRD using a pair of SUBA is given the following result. If \( D = (v, b_1, r_1, k_1, k_{12}, b_1, b_{12}, \lambda_1) \), \( k_i = \sup(k_{12}) \), \( b_1 + b_{12} = b \), and \( r_1 = \frac{c-1}{c}\lambda_i \), and other covariances vanish.

\[ (\lambda - \lambda)^2 \hat{Cov}(b_i, b_j) = (c-1)N\lambda_i[c(v-1)\tau_2 - v\lambda_i\tau_2 + \tau_2^2] \]

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Following Hader and Park (1978) [4], Victorbabu and Narasimham (1991a) [13] and Park and Kim (1992) [7] conditions (2) to (7) give the necessary and sufficient conditions for a measure of slope rotatability for any general second order response surface designs. Further we have,

\[ V(b_i) = \text{equal for } i, \]
\[ V(b_{ij}) = \text{equal for } i, \]
\[ V(b_{ij}) = \text{equal for } i, j \text{, where } i \neq j, \]
\[ \text{Cov}(b_{ij}, b_{ij}) = \text{Cov}(b_{ij}, b_{ij}) = \text{Cov}(b_{ij}, b_{ij}) = 0 \text{ for all } i \neq j, j \neq l, l \neq i. \]

\hfill (9)

Park and Kim (1992) [7] proposed that if the conditions in (2) to (7) and (9) are met, then the following measure \( Q_v(D) \) given below can be used to assess the degree of slope rotatability for any general second order response surface design D with \( v \) independent variables.

\[ Q_v(D) = \frac{1}{2(v-1)\sigma^4} \left[ (v+2)(v+4) \sum_{i=1}^{v} (V(b_i)-\frac{1}{v} \sum_{j=1}^{v} V(b_{ij}))+\frac{(4V(b_i)+\sum_{j=1}^{v} V(b_{ij}))-\frac{1}{v} \sum_{j=1}^{v} (4V(b_i)+\sum_{j=1}^{v} V(b_{ij}))}{v+2} \right]^2 \]
5. Study of measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes

The proposed method of measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes is suggested in this section. Let 

\[ D_1=(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1), \quad k_1=\text{sup}(k_{11},k_{12}), \quad b_{11}+b_{12}=b_1 \]

and 

\[ D_2=(v,b_2,r_2,k_{21},k_{22},b_{21},b_{22},\lambda_2), \quad k_2=\text{sup}(k_{21},k_{22}), \quad b_{21}+b_{22}=b_2 \]

with unequal block sizes respectively. Let \( 2^{(k_1)} \) and \( 2^{(k_2)} \) denote fractional factorial design replicates of \( 2^{k_1} \) and \( 2^{k_2} \) with \( \pm 1 \) levels. The design points achieved from the transpose of incidence matrix of design \( D_1 \) by multiplication (see Das and Narasimham 1962) are denoted by \( [1-(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1)]2^{(k_1)} \). Let \( [1-(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1)]2^{(k_1)} \) be the \( b_1 \) \( 2^{(k_1)} \) design points generated from \( D_1 \) by multiplication. Let \( [\bar{a}-(v,b_2,r_2,k_{21},k_{22},b_{21},b_{22},\lambda_2)]2^{(k_2)} \) be the \( b_2 \) \( 2^{(k_2)} \) design points generated from \( D_2 \) by multiplication. Let \( n_0 \) be the number of central points. Then with the above design points, we can obtain measure of slope rotatability for second order response surface designs using is given in the following theorem.

**Theorem (5.1):** The design points, \( [1-(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1)]2^{(k_1)}U[a-(v,b_2,r_2,k_{21},k_{22},b_{21},b_{22},\lambda_2)]2^{(k_2)}U(n_0) \) give a \( v \)-dimensional measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes in 

\[ N=b_12^{(k_1)}+b_22^{(k_2)}+n_0 \]

design points with level \( \bar{a} \) pre-fixed 

\[ c=\frac{r_12^{(k_1)}+r_22^{(k_2)}a^4}{\lambda_22^{(k_1)}+\lambda_22^{(k_2)}a^4}. \]  

**Proof:** For the design points generated from a pair of SUBA with two unequal block sizes, conditions (2) to (5) are true. Conditions in (2) are true obviously. Conditions (3) to (5) are true as fallows.

\[ \sum x_{im}^2=\bar{a}2^{(k_1)}+r_22^{(k_2)}a^4=3N\lambda_2 \]  
\[ \sum x_{im}^4=\bar{a}2^{(k_1)}+r_22^{(k_2)}a^4=3N\lambda_4 \]  
\[ \sum x_{ju}^2=\lambda_22^{(k_1)}+\lambda_22^{(k_2)}a^4=3N\lambda_4 \]  

Solving equations (12) and (13), we get \( c \).

Measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes can be obtained by solving the given equation (cf. Park and Kim 1992).

\[ c=\frac{r_12^{(k_1)}+r_22^{(k_2)}a^4}{\lambda_22^{(k_1)}+\lambda_22^{(k_2)}a^4}. \]
\[ Q_i(D) = \left[ \sum_{j=1}^{n} \frac{\sum_{j=1}^{X_{ij}}}{N} \left[ 4e^{-V(b_j)} \right]^2 \right] \]

where

\[
(v-1) \left[ 2^{(l_k)} \lambda_n + 2^{(l_k)} b \lambda - 2^{(l_k) + (l_k) a^2} \right] + \\
\left[ (v-1) \left( 2^{(l_k)} + 2^{(l_k)} b \lambda + 2^{(l_k)} \lambda n \lambda - 2^{(l_k)} r^2 \right) \right] a^4 + \\
\left[ (r^2 - \lambda) \left( 2^{(l_k)} b + 2^{(l_k)} b \lambda + 2^{(l_k)} n \lambda - 2^{(l_k)} r^2 \right) \right] \]

\[
e = \left( \frac{2^{(l_k)} (r^2 - \lambda) + 2^{(l_k)} (r^2 - \lambda) a^2}{\lambda_n + 2^{(l_k)} + 2^{(l_k)} b \lambda + 2^{(l_k)} \lambda n \lambda - 2^{(l_k)} r^2 \lambda} \right) \]

\[
Q_v(D) = 4e^{-V(b)} , N
\]

\[
\sum_{j=1}^{n} \frac{\sum_{j=1}^{X_{ij}}}{N} \left[ 4e^{-V(b_j)} \right]^2 ,
\]

(14)

Example: We illustrate the measure of slope rotatability for second order response surface designs, for 12- factors with the help of a pair of SUBA with two unequal block sizes with parameters

\[ D_1 = (v=12, b_1 = 13, r_1 = 4, k_1 = 3, k_2 = 4, b_1 = 4, b_2 = 9, \lambda_1 = 1) \]

\[ D_2 = (v=12, b_2 = 26, r_2 = 6, k_2 = 2, k_1 = 3, b_1 = 6, b_2 = 20, \lambda_2 = 1) \]

The design points, \( [1-(12,13,4,3,4,4,9,1)] \) \( 2^{(l_k) U (a-12,26,6,2,3,6,20,1)] 2^{(l_k) U(n_0=1)} \) will give a measure of slope of rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes in N=417 design points for 12- factors. Here equations (11), (12) and (13) are,

\[
\sum_{j=1}^{n} \frac{\sum_{j=1}^{X_{ij}}}{N} = 64+48a^2 = N\lambda_2
\]

(14)

\[
\sum_{j=1}^{n} \frac{\sum_{j=1}^{X_{ij}}}{N} = 64+48a^4 = cN\lambda_4
\]

(15)

\[
\sum_{j=1}^{n} \frac{\sum_{j=1}^{X_{ij}}}{N} = 16+8a^4 = N\lambda_4
\]

(16)

From equations (15) and (16) we get \( a \approx 2.5100 \), \( \lambda_2 = 0.4424 \), \( \lambda_4 = 0.1592 \) and \( c = 5.5181 \). Non singularity condition (5) is also satisfied as \( 0.8135 > 0.7265 \).

Alternatively from equation (8) we obtain equation (17) directly. Solving (17), we get \( a \approx 2.5100 \), substituting this value of \( a \) in (14), (15) and (16) and on simplification we obtained \( \lambda_2 = 0.4424 \), \( \lambda_4 = 0.1592 \) and \( c = 5.5181 \). Non singularity condition (5) is also satisfied as \( 0.8135 > 0.7265 \).

We exemplify the measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes for \( v = 12 \) factors the design points N=417 hence we get slope rotatability value \( a = 1.5843 \), then the \( Q_v(D) = 0 \). Instead of taking \( a = 1.5843 \), suppose we take \( a = 1.3 \), we get \( c = 5.1763 \), \( e = 0.009371530982 \) and \( Q_v(D) = 2.0235 \times 10^{-6} \). Here \( Q_v(D) \) becomes large it deviates from a second order slope rotatable design.

The following Table 1 gives the values of measure of slope rotatability \( Q_v(D) \) for second order response surface designs using pair of SUBA with two unequal block sizes for \( n_0 = 1, 2, 3, 4, 5 \) and level ‘a’. It can be verified that the measure of slope...
rotatability is zero if and only if a design D is a second order slope rotatable design. Measure of slope rotatability becomes large as D deviates from a second order slope rotatable design.

Table 1: Measure of slope rotatability for second order response surface designs using a pair of symmetrical unequal block arrangements with two unequal block sizes

<table>
<thead>
<tr>
<th>a</th>
<th>n=1 (N=417)</th>
<th>n=2 (N=418)</th>
<th>n=3 (N=419)</th>
<th>n=4 (N=420)</th>
<th>n=5 (N=421)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>7.2455×10^6</td>
<td>1.8700×10^4</td>
<td>2.0231×10^4</td>
<td>1.8139×10^6</td>
<td>1.6347×10^6</td>
</tr>
<tr>
<td>1.3</td>
<td>2.0235×10^6</td>
<td>1.3734×10^4</td>
<td>1.6511×10^4</td>
<td>1.2471×10^4</td>
<td>1.0687×10^4</td>
</tr>
<tr>
<td>1.6</td>
<td>9.2703×10^6</td>
<td>5.8251×10^9</td>
<td>2.4379×10^9</td>
<td>3.3848×10^9</td>
<td>4.4139×10^9</td>
</tr>
<tr>
<td>1.9</td>
<td>1.1008×10^7</td>
<td>1.2247×10^7</td>
<td>1.1149×10^7</td>
<td>1.1212×10^7</td>
<td>1.1270×10^7</td>
</tr>
<tr>
<td>2.2</td>
<td>1.7277×10^7</td>
<td>1.8146×10^7</td>
<td>1.7152×10^7</td>
<td>1.7087×10^7</td>
<td>1.7022×10^7</td>
</tr>
<tr>
<td>2.5</td>
<td>1.9249×10^7</td>
<td>1.9875×10^7</td>
<td>1.9015×10^6</td>
<td>1.8896×10^6</td>
<td>1.8780×10^7</td>
</tr>
<tr>
<td>2.8</td>
<td>1.9535×10^7</td>
<td>7.3812×10^7</td>
<td>1.9258×10^7</td>
<td>1.9120×10^7</td>
<td>1.8984×10^7</td>
</tr>
<tr>
<td>3.1</td>
<td>1.9234×10^7</td>
<td>1.9625×10^7</td>
<td>1.4919×10^7</td>
<td>1.8797×10^7</td>
<td>1.8654×10^7</td>
</tr>
</tbody>
</table>

* Denote the exact slope rotatability values

6. Conclusions

In this paper measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes have been proposed which enable us to assess the degree of slope rotatability for second order response surface designs. It may be pointed out here that the measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes with parameters D_1 = (v=12, b_1=13, r_1=4, k_1=3, k_2=4, b_1=4, b_2=9, λ_1=1), D_2 = (v=12, b_2=26, r_2=6, k_1=2, k_2=3, b_1=20, λ_2=1) has only 471 design points for 12 factors where as corresponding measure of slope rotatability for second order response surface designs using a pair of BIBD with parameters D_1 = (v=12, b_1=33, r_1=11, k_1=4, λ_1=3), D_2 = (v=12, b_2=44, r_1=11, k_2=3, λ_2=1) of Vircotbabu and Surekha (2016) [23] need 881 design points. Similarly, for the measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes with parameters D_1 = (v=15, b_1=33, r_1=9, k_1=5, b_2=9, λ_1=2), D_2 = (v=12, b_2=26, r_2=6, b_1=4, b_2=20, λ_2=1) has only 977 design points for 15 factors where as corresponding measure of slope rotatability for second order response surface designs using a pair of BIBD with parameters D_1 = (v=15, b_1=33, r_1=7, k_1=7, λ_1=3), D_2 = (v=15, b_2=35, r_1=7, k_2=3, λ_2=1) of Vircotbabu and Surekha (2016) [23] need 1241 design points. Thus this new method sometimes lead to measure of slope rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes with lesser number of design points than the measure of slope rotatability for second order response surface design using a pair of BIBD in some cases.

7. References