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Eta-Ricci soliton on W3-Semi symmetric LP Sasakian Manifolfds

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Abstract
In this paper, we study η-Ricci solitons on Lorentzian para-Sasakian manifold satisfying the semi symmetric conditions $R(\xi,X)\bullet W_3(Y,Z)U=0$ and $W_3(\xi,X)\bullet R(Y,Z)U=0$. At the end of this paper, we show that the LP-Sasakian manifold accepting a η-Ricci solitons structure is Einstain.

Keywords: W_3 curvature tensor, W_3 symmetric Sasakian manifold, W_3 semi-symmetric Sasakian manifold and eta-Ricci solitons. AMS 2020 Subject Classification: 53C15, 53C40

Introduction
Ricci-flow is an evolution equation for metric on a Riemannian manifold.
The Ricci-flow equation is given

$$\frac{\partial g}{\partial t} = -2S \text{ [see [1]]} \tag{0.1}$$

on a compact Riemannian manifold M with metric g.
Ricci-soliton is a special solution to the Ricci-flow, but only if it moves by a one parameter family of diffeomorphism and scaling. The Ricci-soliton is given by

$$Lv\,g + 2S + 2\lambda g = 0 \tag{0.2}$$

Where, Lv is Lie derivative in the V direction, S is Ricci curvature tensor, g is a Riemannian metric, V is a vector field and λ is a scalar. η-Ricci soliton is a more general notion of the Ricci-flow. This idea was put forward by J.J Cho and Makoto Kimura [5], and they gave its equation by

$L\xi g + 2S = -2\lambda g - 2\mu\eta\otimes\eta$ λ and μ are constants.

Preliminaries
A Sasakian manifold is a k-contact, but the converse is only true if the dimension n =3. However, a contact metric tensor is Sasakian if and only if

$$R(X,Y)T = g(Y)X - g(X)Y \tag{0.3}$$

In a Sasakian manifold $(M,\phi,\eta,\xi,\lambda,g)$, we can easily see,

$$R(T,X)Y = g(X,Y)T - g(Y)X \tag{0.4}$$

Generally, in n=(2m-1)-dimensional Sasakian Manifold with the structure (ϕ,η,ξ,g), we have

$$R'(X,Y,Z,U) = g(R(X,Y)Z,U = g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \tag{0.5}$$

Where R is the Riemannian curvature tensor of rank $(r) = n-1$
We also observe that the data (g, ξ,λ ,μ), If it sufficiently satisfy equation (0.2), then it is said to be a η-Ricci soliton on the manifold M [2]. More particularly, if we let μ=0, then (g ξ,λ) is a Ricci soliton according to R.S Hamilton [9]. And thus, equation (0.2) is said to be is Shrinking (λ< 0), steady (λ=0) or expanding (λ> 0) [2].

Generalised Lorentzian Para-Sasakian Manifolds

Let M be an n-dimensional smooth manifold, ϕ a tensor field of (1,1)-type, ξ a vector field, η a 1-form and g a Lorentzian metric on M, We say that, (ϕ, ξ, η, g) is a Lorentzian Para-Sasakian structure of M [06] if:

1. $\varphi\xi = 0, \eta \circ \varphi = 0$
2. $\eta(\xi) = -1, \varphi 2 = 1 + \eta \otimes \xi$
3. $g(\varphi^\circ, \varphi^\circ) = g + \eta \otimes \eta$
4. $(\nabla x Y) = g(X,Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X$

for any $X, Y \in \mathfrak{X}(M)$,

From the definition, it follows that η is the g-dual of ξ, that is,

$\eta(X) = g(X, \xi)$for any X ∈ $\mathfrak{X}$(M), ξ then satisfies

(0.6) $g(\xi, \xi) = -1$

Here, φ is a g-symmetric operator, i.e.

$g(\varphi X, Y) = g(X, \varphi Y)$ for any $X, Y \in \mathfrak{X}(M)$.

These structures, (from equation 1-4) have their properties given in the following remark.

Remark 1.1

In [3], and [4], different authors have proved that, On a Lorentzian Para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, for any $X, Y, Z \in \mathfrak{X}(M)$, the following relations holds:

$$\nabla x\, \xi = \varphi X$$
$$\eta(\nabla x\,) = 0, \nabla \xi \xi = 0$$
$$R(X,Y)\xi = -\eta(X)Y + \eta(Y)X$$
$$\eta\,(R\,(X,Y)\,\xi) = 0 (\nabla x Y) = (\nabla Y X) = g(\varphi X, Y), \nabla x \eta = 0$$

And

$$L\xi\varphi = 0, L\xi\eta = 0, L\xi g = 2g\,(\varphi^\circ, {}^\circ)$$

Where R is the Riemannian Curvature tensor field, and ∇, the Levi-Civita associated to g.
The proofs of these properties are given by Adara [1]

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U.C De and N. Guha [10] gave the definition of semi-symmetric as $R(X,Y).R(Z,U) = 0$
On the same line, we can also have,

Definition 2.1: A Sasakian manifold is said to be W_3-semi-symmetric if

$R(\xi, X).W_3(Y,Z) = 0$ (2.1)

Theorem 2.2: If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the manifold M_n,and if (g,ξ,λ,μ) is a η- Ricci soliton on M_n, and $R(\xi, X).W_3(Y,Z) = 0$, then λ=1, when μ=n.

Proof

The condition that W_3 must satisfy is given by,

$W_3(R(\xi,X)Y,Z) + W_3(Y, R(\xi,X)Z = 0$ (2.2)

For $X, Y, Z \in \mathfrak{X}(M)$

Using the definition (0.4) in LP-Sasakian, we obtain η-Ricci Soliton (eqn 0.1).
Now, using the relation,

$R(\xi,X)Z = g(X,Z)\xi - \eta(Z)X$ (2.3)

We have,

$W_3 X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)\phi X - Ric(X,Z)Y]$ (2.4)

Where $\phi X = (n-1)X$

Thus, $S(X,Y) = g(\phi X, Y) = (n-1)g(X,Y)$ (2.5)

With the above conditions, we now compute each term in equation (2.2) separately, taking inner-product with respect to U and T respectively, and using (1.1) and (2.5) to obtain,
First term,

$W_3(R(\xi,X)Y,Z,U,T)g(Z,U)g(g(X,Y)\xi - \eta(Y)X,T) - g(Z,T)g(g(X,Y)\xi - \eta(Y)X,U) + \frac{1}{n-1}[g(Z,U)Ric(R(\xi,X)Y,T) - Ric(R(\xi,X)Y,U)g(Z,T)]$ (2.6)

putting $X = Y = T = \xi$, the results follows $W_3(R(\xi,X)Y,Z,U,T) = 0$ (2.6)

The computation of the second term also yield,

$W_3(Y,R(\xi,X)Z,U,T)(Y,R(\xi,X)Z,U,T)+\frac{1}{n-1}[g(R(\xi,X)Z,U)Ric(Y,T)-g(R(\xi,X)Z,T)Ric(Y,U)= g(Y,T)g(g(X,Z)\xi-g(\xi,Z)X,U)-g(Y,U)g(g(X,Z)\xi-g(\xi,Z)X,T)+\frac{1}{n-1}[g(R(\xi,X)Z,U)Ric(Y,T)-g(R(\xi,X)Z,T)Ric(Y,U)$ (2.7)

Similarly, putting X=Y=T=ξ in (2.7) we obtained,

$W_3(Y,R(\xi,X)Z,U,T)=Ric(Y,\xi)\{\frac{1}{n-1}[-\eta(X)+\eta(X)-(-\eta(X)+\eta(X)]$ (2.8)

$\Rightarrow Ric(Y,\xi)(0)=0$

But $Ric(Y,\xi)\neq 0$

We know from (2.5)

$Ric(Y,\xi)=(n-1)\eta(Y)$ (2.9)

But η-Ricci Soliton in LP-Sasakian manifold, we see that

$Ric(X,Y)=S(X,Y)=g(\varphi Y,X)-\lambda g(X,Y)-\mu\eta(X)\eta(Y)$ (2.10)

Putting X=ξ $Ric(\xi,Y)=(\mu-\lambda)\eta(Y)$ (2.11)

Solving equation (2.9) and (2.11) simultaneously,

We observe

$\mu-\lambda=n-1$

So that when $\lambda=1$ the $\mu=n$

Hence the theorem.

Corollary 2.3: If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the Manifold M_n, (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if $R(\xi,X).W_3(Y,Z)=0$, then (M_n,g) is Einstein Manifold

Theorem 2.4: If (μ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the manifold M_n, and if (g,ξ,λ,μ) is a η- Ricci soliton on M_n, and $S(\xi,X).W_3(Y,Z)=0$, then λ=1, when μ=n.

Proof:

If the Sasakian space is a W_3 −semi-symmetric, then $S(\xi,X).W_3(Y,Z)=0$

And the condition that W_3 must satisfy is given by,

$S(X,W_3(Y,Z)U)\xi-S(\xi,W_3(Y,Z)U)X+S(X,Y)W_3(\xi,Z)U- S(\xi,Y)W_3(X,Z)U+S(X,Z)W_3(Y,\xi)U-S(\xi,Z)W_3(Y,X)U+S(X,U)W_3(Y,Z)\xi-S(\xi,U)W_3(Y,Z)X=0$ (2.13)

For X,Y,Z,U ∈ 𝔛(M)

Now taking inner product with respect to ξ, our equation (2.13) becomes

$-S(X,W_3(Y,Z)U-S(\xi,W_3(Y,Z)U\eta(X)+S(X,Y)\eta(W_3(\xi,Z)U)- S(\xi,Y)\eta(W_3(X,Z)U+S(X,Z)\eta(W_3(Y,\xi)U-S(\xi,Z)\eta(W_3(Y,X)U)+S(X,U)\eta(W_3(Y,Z)\xi-S(\xi,U)\eta(W_3(Y,Z)X=0$ (2.14)

From (2.5), we have $S(\xi,\xi)=\mu-\lambda$. (2.15)

Now, we observed that equation (2.14) has eight terms. We computed each term independently and when subjected to certain equivalent conditions, we obtained, From the first term, using (2.10),

$S(X,W_3(Y,Z)U=-g(\varphi X,W_3(Y,Z)U)-\lambda g(X,W_3(Y,Z)U)- \mu\eta(X)\eta(W_3(Y,Z)U)$ (2.16)

From Pokhariyal's definition of W_3[7], and putting U=Z=ξ, we got

$W_3(Y,\xi)\xi=R(Y,\xi)\xi+\frac{1}{n-1}[\eta(\xi)\phi Y-Ric(Y,\xi)\xi]$ (2.16)

This then implies that,

$g(X,W_3(Y,\xi)\xi)=-g(X,Y)-\eta(X)\eta(Y)-\frac{1}{n-1}[Ric(X,Y)+ \eta(X)Ric(Y,\xi)]$ (2.07)

Substituting X=Y=ξ into (2.17) and using (0.6), we have

$\eta(W_3(Y,\xi)\xi)=0$ (2.18)

Also, using (2.5), $R(\xi,\xi)=\lambda-\mu$

Since $g(Y,\xi)$ is vanishing, we then obtained

$g(\phi X,W_3(Y,\xi)\xi)-g(\varphi X,Y)\left[1+\frac{1}{n-1}\right]+g(\varphi X,Y)\left[1+\frac{1}{n-1}\right]+\lambda g(X,Y)+\eta(X)\eta(Y)+\frac{1}{n-1}[Ric(X,Y)+(\lambda-\mu)\eta(X)$ (2.19)

Now, computing the second term, we obtained

$S(\xi,W_3(Y,Z)U)=-\lambda\eta(W_3(Y,Z)U)+\mu\eta(W_3(Y,Z)U)$ (2.20)

Again, from Pokhariyal's definition of W_3[8], and by putting U=Z=ξ into (2.20), we have
$\eta(W_3(Y,\xi)\xi) = 0$ (2.21)

Computing the third term, we see that
$g(\varphi X, W_3(Y,\xi)\xi) = -g(\varphi X, Y) + \frac{1}{n-1}[g(X,Y) + \eta(X)\eta(Y) + \lambda g(\varphi X, Y)].$ (2.22)

Next, we combined the equations (2.19), (2.21) and (2.22) and obtained
$S(X, W_3(Y,\xi)\xi) = -g(\varphi X, Y)\left[1 + \frac{\lambda}{n-1}\right] + \{\left[\lambda - \frac{1}{n-1}\right](g(X,Y) + \eta(X)\eta(Y)\} = 0$ (2.23)
Computing the fourth term, putting U=Z=ξ, $\eta(W_3(X,Z)U)$ became
$\eta(W_3(X,\xi)\xi) = \frac{2}{n-1}(\lambda + \mu)\eta(X)$ (2.24)

Using (2.11), we see that,
$S(\xi, Y)\eta(W_3(X,\xi)\xi) = \frac{2}{n-1}(\lambda + \eta)(\mu - \lambda)\eta(X)\eta(Y)$ (2.25)

Computation of the fifth term, using (2.11) with U=ξ, gave
$\eta(W_3(Y,\xi)\xi) = \frac{2}{n-1}(\lambda + \mu)\eta(X)$ (2.26)

This then implied that
$S(X,\xi)\eta(W_3(Y,\xi)\xi) = \frac{2}{n-1}(\lambda + \eta)(\mu - \lambda)\eta(X)\eta(Y)$ (2.27)

Observe that equations (2.25) and (2.27) cancels out since one is negative and the other positive.
Similar computations and conditions led to the sixth and seventh term vanishing as well.
We finally computed the eighth term as follows;
Consider, $S(\xi, U)\eta(W_3(Y,Z)X$, setting Z=U=ξ and using (2.11)
We also considered $\eta(W_3(Y,Z)X$, setting Z=ξ
Then we obtained
$\eta(W_3(Y,\xi)X) = \eta(\eta(X)Y) - g(X,Y)\xi - \frac{1}{n-1}[\eta(X)Ric(Y,\xi) + Ric(Y,X)]$ (2.28)

Also
$g(X, W_3(Y,\xi)\xi) = -\eta(Y) + \eta(Y) + \frac{1}{n-1}[-Ric(\xi,Y) + Ric(Y,\xi)]$ (2.29)

This led to
$g(\varphi X, W_3(Y,\xi)\xi) = g(\varphi X, Ric(Y,\xi)\xi) + \frac{1}{n-1}[\eta(\xi)g(\varphi X, \varphi Y) - Ric(Y,\xi)g(\varphi X,\xi)]$ (2.30)

Putting X=Z=U=ξ, $S(\xi,U)\eta(W_3(Y,Z)X$ became $S(\xi,\xi)\eta(W_3(Y,\xi)\xi$
Now, using (2.11), and the fact that $-g(\varphi\xi,\xi) = 0$ *and* $S(Y,\xi) = (n-1)\eta(Y)$
In LP-Sasakian, we easily see that $\lambda - \mu = n - 1$

Alternatively,
Since
$\eta(W_3(Y,Z)X) = \eta(R(Y,Z)X) + \frac{1}{n-1}[g(Z,X)g(\varphi Y,\xi) - Ric(Y,Z)g(Z,\xi)$ (2.31)

Putting X=Z=U=ξ, and using $\eta(\xi) = -1$
We obtained
$\eta(W_3(Y,\xi)\xi) = \eta(\eta(\xi)Y - \eta(Y)\xi) = 0$ (2.32)

But we know from (2.15) that
$S(\xi,\xi) = \lambda - \mu$ (2.33)

Hence, we conclude from LP-Sasakian manifold that
$S(Y,\xi) = (n-1)\eta(Y)$ (2.34)

Finally, solving (2.33) and (2.34) simultaneously, we obtained
$\mu - \lambda = n - 1$

And thus, whenever μ=n, then λ=1. Hence the theorem.

Corollary 2.5: If (μ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the Manifold M_n, (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if $S(\xi,X).W_3(Y,Z) = 0$, then (M_n,g) is Einstein Manifold

Discussion

In LP-Sasakian manifold, W_3 curvature tensor satisfies semi-symmetric and cyclic properties with fixed X [8]. The properties are similar to those of Weyl's projective tensor. Therefore, the two tensors can be used alternatively to study the physical and geometrical characteristics of manifolds.

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