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A study of eta- Ricci soliton on W_5-semi symmetric LP sasakian manifolfds

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Abstract
In this paper, we study η-Ricci solitons on Lorentzian para-Sasakian manifold satisfying R(ξ,X)•W_5(Y,Z)U=0 and W_5(ξ,X)•R(Y,Z)U=0 conditions.
We prove that on a Lorentzian para-Sasakian manifold (M,ξ,η,g), the Ricci curvature tensor satisfying any one of the given conditions, the existence of η-Ricci soliton then implies that (M,g) is Einstein manifold. We also conclude that in these cases, there is no Ricci soliton on M, with the potential vector field ξ (the killing vector).

Keywords: W_5 curvature tensor, W_5 symmetric Sasakian manifold, W_5 semi-symmetric Sasakian manifold and eta-Ricci solitons. *AMS 2020 Subject Classification: 53C15, 53C40.*

Introduction
Ricci-flow is an evolution equation for metric on a Riemannian manifold. It defines a kind of non-linear diffusion equation similar to that of heat equation for metric under Ricci-flow.
The Ricci-flow equation is given
(0.1) $\frac{\partial g}{\partial t} = -2S$

On a compact Riemannian manifold M with metric g. Ricci-soliton is a similar solution to the Ricci-flow, but only if it moves by a one parameter family of diffeomorphism and scaling. The Ricci-soliton has its equation given by
(0.2) $L_v g + 2S + 2\lambda g = 0$

Where, L_v is Lie derivative in the V direction, S is Ricci curvature tensor, g is a Riemannian metric, V is a vector field and λ is a scalar. η-Ricci soliton is a more general notion of the Ricci-flow. This idea was put forward by J.J Cho and Makoto Kimura [05], and they gave its equation by
$L_\xi g + 2S = -2\lambda g - 2\mu\eta\otimes\eta$
λ and μ are constants.

1. Preliminaries
A Sasakian manifold is a k-contact, but the converse is only true if the dimension n =3. However, a contact metric tensor is Sasakian if and only if
(0.3) R(X, Y)T= g(Y)X-g(X)Y

In a Sasakian manifold (M,ϕ, η, ξ,λ, g), we can easily see,
(0.4) R(T, X)Y= g(X,Y)T-g(Y)X

Generally, in n=(2m-1)-dimensional Sasakian Manifold with the structure (ϕ, η, ξ, g), we have
(0.5) $R'(X, Y, Z, U)$=g(R(X,Y)Z,U =g(Y, Z)g(X,U)-g(X,Z)g(Y,U)

Where R is the Riemannian curvature tensor of rank(r)=n-1
We also observe that the data (g, ξ,λ, μ), If it sufficiently satisfy equation (0.2), then it is said to be a η-Ricci soliton on the manifold M[02]. More particularly, if we let μ=0, then (g ξ,λ) is a

Ricci soliton according to R.S Hamilton [11]. And thus, equation (0.2) is said to be is shrinking, steady or expanding according to the value of λ [02]

Generalised lorentzian para-sasakian manifolds
Let M be an n-dimensional smooth manifold, ϕ a tensor field of (1, 1)-type, ξ a vector field, η a 1-form and g a Lorentzian metric on M, We say that, (ϕ,ξ,η,g) is a Lorentzian Para-Sasakian structure of M[06] if:

1. $\varphi\xi = 0$, $\eta\circ\varphi = 0$
2. $\eta(\xi) = -1$, $\varphi^2 = 1 + \eta\otimes\xi$
3. $g(\varphi^{\circ},\varphi^{\circ}) = g + \eta\otimes\eta$
4. $(\nabla_x Y) = g(X,Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X$

For any X, Y ∈ 𝔛(M),

From the definition, it follows that η is the g-dual of ξ, that is, $\eta(X) = g(X,\xi)$
For any X ∈ 𝔛(M), ξ then satisfies
(0.6) $g(\xi,\xi) = -1$

Here, φ is a g-symmetric operator, i.e.
$g(\varphi X, Y) = g(X, \varphi Y)$
For any X, Y ∈ 𝔛(M).

These structures, (from equation 1-4) have their properties given in the following remark.

Remark 1.1
In [03], and [04], different authors have proved that, On a Lorentzian Para-Sasakian manifold (M,φ,ξ,η,g), for any X,Y,Z ∈ 𝔛(M), the following relations holds:

5. $\nabla_x \xi = \varphi X$
6. $\eta(\nabla_x) = 0$, $\nabla_\xi\xi = 0$
7. $R(X,Y)\xi = -\eta(X)Y + \eta(Y)X$
8. $\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z)$, $\eta(R(X, Y)\xi) = 0$
9. $(\nabla_x Y) = (\nabla_Y X) = g(\varphi X, Y)$, $\nabla_x\eta = 0$
10. $L_\xi\varphi = 0$, $L_\xi\eta = 0$, $L_\xi g = 2g(\varphi^\circ, {}^\circ)$

Where R is the Riemannian Curvature tensor field, and ∇, the Levi-Civita associated to g.
The proofs of these properties are given by Adara [01]

2. Eta-Ricci Soliton on W_5-Semi Symmetric LP Sasakian Manifolfds.

U.C De and N. Guha [12] gave the definition of semi-symmetric as $R(X,Y)R(Z,U) = 0$
On the same line, we can also have,

Definition 2.1: A Sasakian manifold is said to be W_5-semisymmetric if
(2.1) $R(\xi,X)W_5(Y,Z) = 0$

Theorem 2.2: If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the manifold M_n,and if (g,ξ,λ,μ) is a η- Ricci soliton on M_n, and $R(\xi,X).W_5(Y,Z)=0$, then λ=1, when μ=n.

Proof
If the Sasakian space is a W_5 −semi-symmetric, then $R(\xi,X)W_5(Y,Z) = 0$

And the condition that W_5 must satisfy is given by,
(2.2) $W_5(R(\xi,X)Y,Z) + W_5(Y,R(\xi,X)Z = 0$
For X, Y, Z, U ∈ 𝔛(M)

Pokhariyal and Mishra [7] gave the definition of W_5 as
(2.3) $W_5(X,Y,Z,U) = R(X,Y.Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,U)Ric(X,Z)$

Now we compute the two terms on the R.H.S of (2.3) as follows:
First term,
(2.4) $W_5(R(\xi,X)Y,Z,U,\xi) = R(R(\xi,X)Y,Z,U,\xi) + \frac{1}{n-1}[g(R(\xi,X)Y,U)Ric(Z,\xi) - g(R(Z,\xi)Ric(R(\xi,X)Y,U)]$

Upon expansion of the three terms on the R.H.S of (2.4) independently, we obtained,
For the first term, using (8) and putting X=Y=U=ξ, the results follows
(2.5) $R(R(\xi,X)Y,Z,U,\xi) = R(g(X,Y)\xi - \eta(Y)X,Z,U,\xi)=0$

When we put X=Y=U=ξ and then use (0.6) in the computation of the second term, we obtained
(2.6) $g(R(\xi,X)Y,U)Ric(Z,\xi) = g(g(X,Y)\xi - g(\xi,Y)X,U)Ric(Z,\xi) = 0$

With similar conditions, the computation of the third term also yield,
(2.7) $g(Z,\xi)Ric(R(\xi,X)Y,U) = \eta(Z)Ric(g(X,Y)\xi - g(\xi,Y)X,U) = 0$

Computation of the second term of (2.2) gave,
(2.8) $W_5(Y,R(\xi,X)Z,U,\xi) = R(Y,R(\xi,X)Z,U,\xi) + \frac{1}{n-1}[g(Y,U)Ric(R(\xi,X)Z,\xi) - g(R(\xi,X)Z,\xi)Ric(Y,U)]$

We observed that equation (2.8) has 3-terms on the R.H.S. Their expansion (independently) leads to:
From the first term, putting U=Z=ξ, and then using (2) and (0.6), we obtained
(2.9) $R(Y,R(\xi,X)Z,U,\xi) = g(Y,\xi)g(R(\xi,X)Z,U) - g(R(\xi,X)Z,\xi)g(Y,U) = 0$

Expanding the second term and putting the same conditions as in (2.9), we have
(2.10) $g(Y,U)Ric(\xi,X)Z,\xi) = g(Y,U)Ric(g(X,Z)\xi - \eta(Z)X,\xi) = 0$

And lastly, the third term gave,
(2.11) $-Ric(Y,U)g(R(\xi,X)Z,\xi) = -Ric(Y,U)\{g(g(X,Z)\xi - \eta(Z)X,\xi)\} = Ric(Y,U)\{g(X,Z) + \eta(X)\eta(Z)$

We see clearly that our subsequent expansions, with the necessary conditions, left only (2.11) non-vanishing. With simplification and re-substitution into (2.2), we could see,

If we put X=Y=ξ into (2.11), the we have;
$Ric(\xi,U)\{\eta(Z)-\eta(Z)\}=0$

But we know $Ric(\xi,U)\neq 0$
And from LP-Sasakian,
(2.12) $Ric(\xi,U)=(n-1)\eta(U)$

We also have, from η-Ricci soliton,
(2.13) $S(X,Y)=Ric(X,Y)=g(\varphi X,Y)-\lambda g(X,Y)-\mu\eta(X)\eta(Y)$

Setting Y=U and X=ξ
(2.14) $Ric(\xi,U)=-0-\lambda\eta(U)+\mu\eta(U)=(-\lambda+\mu)\eta(U)$

Solving (2.12) and (2.14) simultaneously,
$\Rightarrow \mu-\lambda=n-1$

Thus, it follows that, When μ=n, then λ=1.
Hence the theorem.

Corollary 2.3: If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the Manifold M_n, (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if R(ξ,X).W_5(Y,Z)=0, then (M_n,g) is Einstein Manifold

Theorem 2.4: If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the manifold M_n, and if (g,ξ,λ,μ) is a η- Ricci soliton on M_n, and W_5 (Y,Z). R(ξ,X)=0, then λ=1, when μ=n.

Proof
If the Sasakian space is a W_5 −semi-symmetric, then $R(\xi,X)W_5(Y,Z)=0$

And the condition that W_5 must satisfy is given by,
(2.13) $W_5(X,R(Y,Z)U)\xi-W_5(\xi,R(Y,Z)U)X+W_5(X,Y)R(\xi,Z)U-\ W_5(\xi,Y)R(X,Z)U+W_5(X,Z)R(Y,\xi)U-W_5(\xi,Z)R(Y,X)U+\ W_5(X,U)R(Y,Z)\xi-W_5(\xi,U)R(Y,Z)X=0$

We observe that (2.13) has eight terms.
On expanding each term independently, then taking inner product with respect to ξ, we obtained
(2.14) $W_5(X,R(X,Z)U,\xi,T)=R(X,R(Y,Z)U,\xi,T)+\frac{1}{n-1}[g(X,\xi)Ric(R(Y,Z)U,T)-g(R(Y,Z)U,T)Ric(X,\xi)]$

Putting T=ξ into (2.14)
(2.15) $W_5(X,R(Y,Z)U,\xi,\xi)=\eta(X)\{g(Z,U)\eta(Y)-\eta(Z)g(Y,U)\}-\eta(X)\{g(Z,U)\eta(Y)-g(Y,U)\eta(Z)+\frac{1}{n-1}[\eta(X)Ric(R(Y,Z)U,\xi)-g(R(Y,Z)U,\xi)Ric(X,\xi)]=0$

Computing the second term, putting X=Y=T=ξ, we get
(2.16) $W_5(\xi,R(\xi,Z)U,\xi,\xi)=\{-\big(-g(Z,U)-\eta(U)\eta(Z)+\big(-g(Z,U)-\ \eta(U)\eta(Z)\big)\big\}+\frac{1}{n-1}[-(n-1)g(R(\xi,Z)U,\xi)+(n-1)g(R(\xi,Z)U,\xi)]=0\Rightarrow W_5(\xi,R(\xi,Z)U,X,\xi)=0$

Computing the third term, and putting X=Y=T=ξ we obtained
(2.17) $W_5(\xi,\xi,R(\xi,Z)U,\xi)=\{-(g(\xi,g(Z,U)\xi-g(U,\xi)Z)+(g(\xi,g(Z,U)\xi-\ g(\xi,U)Z)\}+\frac{1}{n-1}[-(n-1)g(R(\xi,Z)U,\xi)+(n-1)g(R(\xi,Z)U,\xi)]$
$=0$

Also, the fourth term with similar conditions yield
(2.18) $W_5(\xi,\xi,R(\xi,Z)U,\xi)=-\big(-g(Z,U)-\eta(U)\eta(Z)\big)+\big(-g(Z,U)-\ \eta(U)\eta(Z)\big)+\frac{1}{n-1}[g(\xi,R(X,Z)U)Ric(Y,T)-g(Y,T)Ric(\xi,R(X,Z)U)]=0$
Computation of the fifth term yielded the following results
(2.19) $W_5(X,Y,R(Y,\xi)U,T)=R(X,Z,R(Y,\xi)U,T)+\ \frac{1}{n-1}[g(X,R(Y,\xi)U)Ric(Z,T)-g(Z,T)Ric(X,R(Y,\xi)U)]$

Now putting X=Y=T=ξ into (2.19), we obtained
(2.20) $W_5(X,Z,R(Y,\xi)U,T)=2\eta(Z)\eta(R(\xi,\xi)U)$

From the definition of Ricci Curvature tensor, (2.21) $R(Y,\xi)U = \eta(U)Y - g(Y,U)\xi$

Putting Y=ξ into (2.21)
(2.22) $\Rightarrow R(\xi,\xi)U = (\eta(U)\xi - \eta(U)\xi) = 0$

We proceed to compute the sixth term and obtained
(2.23) $W_5(\xi,Y,R(Y,X)U,T) = R(\xi,Z,R(Y,X)U,T) + \frac{1}{n-1}[g(\xi,R(Y,X)U)Ric(Z,T) - g(Z,T)Ric(\xi,R(Y,X)U)]$

Putting X=Y=T=ξ into (2.23), we obtained
(2.24) $W_5(\xi,Z,R(Y,\xi)U,T) = 2\eta(Z)\eta(R(\xi,\xi)U)$

From (2.21) and with Y=ξ, we also see that,
(2.25) $R(\xi,\xi)U = (\eta(U)\xi - \eta(U)\xi) = 0$

Hence, we can easily conclude that,
$W_5(\xi,Z,R(Y,\xi)U,T) = 0$

Computing the seventh term,
(2.26) $W_5(X,U,R(Y,Z)\xi,T) = R(X,U.R(Y,Z)\xi,T) + \frac{1}{n-1}[g(X,R(Y,Z)\xi)Ric(U,T) - g(U,T)Ric(X,R(Y,Z)\xi)]$

Putting X=Y=T=ξ into (2.26) we obtained
(2.27) $W_5(X,U,R(Y,Z)\xi,T) = -g(U,Z) + \eta(U)\eta(Z)$

We now compute the last term of equation (2.13)
(2.28) $W_5(\xi,U,R(Y,Z)X,T) = R(\xi,U,R(Y,Z)X,T) + \frac{1}{n-1}[g(\xi,R(Y,Z)X)Ric(U,T) - g(U,T)Ric(\xi,R(Y,Z)X)]$

From the computation of the 7^{th} and 8^{th} terms, we see that (2.27) and (2.28) do not vanish. Summing the up and putting X=Y=T=ξ we obtained
(2.29) $-\frac{1}{n-1}[(n-1)\eta(U)g(\xi,R(Y,Z)X) - \eta(U)Ric(\xi,R(Y,Z)X)] = 0$

Since $\eta(U) = \eta(T) = \eta(\xi) = -1$
Then,
(2.30) $-(n-1)g(\xi,R(Y,Z)X) + Ric(\xi,R(Y,Z)X) = 0$
$\Rightarrow Ric(\xi,R(Y,Z)X) = (n-1)g(\xi,R(Y,Z)X)$

Or, From the definition of Ricci-soliton,
(2.30) $Ric(\xi,U) = (n-1)g(\xi,U)$

But η-Ricci soliton in LP-Sasakian is given by
(2.31) $S(X,Y) = Ric(X,Y) = -g(\varphi X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y)$

Putting X=ξ, into (2.31)
(2.32) $S(\xi,Y) = -\lambda\eta(Y) + \mu\eta(Y) = (\mu - \lambda)\eta(Y)$

Finally, solving (2.30) and (2.32) simultaneously, we obtained
$\mu - \lambda = n - 1$

And thus, whenever μ=n, then λ=1. Hence the theorem.

Corollary 2.5: If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the Manifold M_n, (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if W_5(Y,Z).R(ξ,X) =0, then (M_n,g) is Einstein Manifold

Discussion
In LP-Sasakian manifold, W_5 tensor satisfies properties some of which are similar to those of Weyl's projective tensor. Therefore, the two tensors can be used alternatively to study the physical and geometrical characteristics of manifolds.
Furthermore, from the above results, we conclude that, except for the case where the potential vector field ξ is vanishing, we can use W_5 in various physical and geometrical spheres in place of Riemannian curvature tensors.

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