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Second order rotatable designs under tri-diagonal correlated structure of errors using a pair of symmetrical unequal block arrangements with two unequal block sizes

K Raghavendra Swamy and B Re Victorbabu

Abstract
In this paper, a study on second order rotatable designs under tri-diagonal correlated structure of errors using a pair of symmetrical unequal block arrangements with two unequal block sizes is suggested and the variance of the estimated response for different values of the tri-diagonal correlated coefficient (P) and the distance from the centre (d) for $6 \leq v \leq 15$ (V -factors) are studied.

Keywords: Second order rotatable designs, tri-diagonal correlated errors, Symmetrical unequal block arrangements with two unequal block sizes

1. Introduction

Response surface methodology (RSM) often deals with a natural and desirable property rotatability, which requires that, the variance of the predicted response at a point remains constant at all such points that are equidistant from the design center. To achieve stability in prediction variance, this important property of rotatability was developed.

The concept of rotatability, which is very important in response surface designs was introduced by Box and Hunter [1]. Das and Narasimham [2] constructed second order rotatable designs (SORD) through balanced incomplete block designs (BIBD). Raghavarao [10] introduced symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Raghavarao [11] constructed SORD using incomplete block designs. Narasimham *et al.* [3], Panda and Das [4] introduced first order rotatable designs with correlated errors. Das [5, 6, 7, 8] introduced and studied robust SORD.

Rajayalakshmi [12] suggested some contributions to second order rotatable and slope rotatable designs under different correlated error structures. Rajyalakshmi and Victorbabu [13] suggested an empirical study of SORD under tri-diagonal correlated structure of errors using central composite designs. Rajyalakshmi and Victorbabu [14] suggested an empirical study of SORD under tri-diagonal correlated structure of errors using incomplete block designs. Raghavendra Swamy and Victorbabu [15] suggested an empirical study on SORD under tri-diagonal correlated structure of errors using a pair of balanced incomplete block designs.

In this paper, a study on second order rotatable designs under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes is suggested and the variance of the estimated response for different values of the tri-diagonal correlated coefficient (ρ) and the distance from the centre (d) for v factors $6 \leq v \leq 15$ are studie.

2. Conditions for Second Order Rotatable Designs under Tri-diagonal Correlated Structure of Errors

A second order response surface design $D=((x_{iu}))$ for fitting,

$$Y_u(x)=b_0+\sum_{i=1}^{v} b_i x_{iu}+\sum_{i=1}^{v} b_{ii} x_{iu}^2+\sum_{i=1}^{v}\sum_{i<j=1}^{v} b_{ij} x_{iu} x_{ju}+e_u \quad (1)$$

where x_{iu} denotes the level of the i^{th} factor $(i=1,2,....,v)$ in the u^{th} run $(u=1,2,....,N)$ of the experiment, e_u's are correlated random errors is said to be a SORD under tri-diagonal correlated error of structure, if the variance of the estimated response of $\hat{Y}_u$ from the fitted surface is only a function of the distance, ($d^2=\sum_{i=1}^{v} x_i^2$) of the point ($x_{1u},x_{2u},...,x_{vu}$) from from the origin (centre) of the design. Such a spherical variance function for estimation of responses in the second order response surface is achieved if the design points satisfy the following conditions [cf. Das, [5, 6, 7, 12, 15]].

Let ρ be the correlation between errors of any two observations, each has the same variance σ^2 then the necessary and sufficient conditions for second order rotatability under the tri-diagonal variance - covariance structure are

$$\sum_{u=1}^{2n} X_{iu}=0,\ \sum_{u=1}^{2n} X_{iu}X_{ju}=0,\ \sum_{u=1}^{2n} X_{iu}X_{ju}^{2}=0,\ \sum_{u=1}^{2n} X_{iu}X_{ju}X_{ku}=0,\ \sum_{u=1}^{2n} X_{iu}^{3}=0$$

$$\sum_{u=1}^{2n} X_{iu}X_{ju}^{3}=0,\ \sum_{u=1}^{2n} X_{iu}X_{ju}X_{ku}^{2}=0,\ \sum_{u=1}^{2n} X_{iu}X_{ju}X_{ku}X_{lu}=0 \text{ for } i\neq j\neq k\neq l \quad (2)$$

$$\sum_{u=1}^{2n} X_{iu}^{2}=\text{constant}=2n\gamma_2, \text{ for all } i, \quad (3)$$

$$\sum_{u=1}^{2n} X_{iu}^{4}=\text{constant}=3(2n)\gamma_4, \text{ for all } i, \quad (4)$$

$$\sum_{u=1}^{2n} X_{iu}^{2}X_{ju}^{2}=\text{constant}=2n\gamma_4, \text{ for all values } i\neq j \quad (5)$$

From (4) and (5), we have

$$\sum_{u=1}^{2n} X_{iu}^{4}=3\sum_{u=1}^{2n} X_{iu}^{2}X_{ju}^{2} \quad (6)$$

where γ_2 and γ_4 are constants. The summation is over the design points, and ρ is the correlation coefficient.

The Variances and Covariances of the estimated parameters under the tri-diagonal model are as follows:

$$V(\hat{b}_0)=\frac{\gamma_4(v+2)(1+\rho)\sigma^2}{2n[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]}$$

$$V(\hat{b}_i)=\frac{\sigma^2(1-\rho^2)}{2n\gamma_2}$$

$$V(\hat{b}_{ij})=\frac{\sigma^2(1-\rho^2)}{2n\gamma_4}$$

$$V(\hat{b}_{ii})=\frac{\sigma^2(1-\rho^2)[\gamma_4(v+1)-(v-1)\gamma_2^2(1-\rho)]}{2(2n)\gamma_4[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]}$$

$$Cov(\hat{b}_0,\hat{b}_{ii})=\frac{-\gamma_2\sigma^2(1-\rho^2)}{2n[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]}$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj})=\frac{\sigma^2(1-\rho^2)[\gamma_2^2(1-\rho)-\gamma_4]}{2(2n)\gamma_4[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]} \quad (7)$$

and other covariances are zero.

An inspection of the variance function $\hat{b}_0$ shows that a necessary condition for the existence of a non-singular second order design

$$\frac{\gamma_4}{\gamma_2^2}>\frac{v(1-\rho)}{(v+2)}$$

(8) (Non- singularity condition)

The variance of the response $\hat{Y}_u$ at any point estimated through the Surface comes out as,

$$V(\hat{Y}_u)=V(\hat{b}_0)+d^2[V(\hat{b}_i)+2\,cov(\hat{b}_0,\hat{b}_{ii})]+d^4\,V(\hat{b}_{ii}) \quad (9)$$

Hence the variance of estimate of $\hat{Y}_u$ becomes,

$$V(\hat{Y}_u)=\frac{[\gamma_4(v+2)(1+\rho)\sigma^2]}{2n[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]}+\left[\frac{\sigma^2(1-\rho^2)}{2n\gamma_2}+2\left(-\frac{\gamma_2\sigma^2(1-\rho^2)}{2n[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]}\right)\right]d^2+\frac{\sigma^2(1-\rho)[\gamma_4(v+1)-(v-1)\gamma_2^2(1-\rho)]}{2(2n)\gamma_4[\gamma_4(v+2)-v\gamma_2^2(1-\rho)]}d^4 \quad (10)$$

3. Method of construction of SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes

Following Raghavarao [11] and Das [7], Rajyalakshmi and Victorbabu [14] methods of constructions of SORD and RSORD, an empirical study on SORD under tri-diagonal correlated structure of errors using pair of SUBA with two unequal block sizes is studied. Let ρ be the correlation errors of any two observations.

Define $D_1=(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1)$, $k_1=\sup(k_{11},k_{12})$, $b_{11}+b_{12}=b_1$ with $r_1 \lesseqgtr 3\lambda_1$ and $D_2=(v,b_2,r_2,k_{21},k_{22},b_{21},b_{22},\lambda_2)$, $k_2=\sup(k_{21},k_{22})$, $b_{21}+b_{22}=b_2$ with $r_2 \gtreqless 3\lambda_2$ be two SUBA with two unequal block sizes respectively. Let $2^{t(k_1)}$ and $2^{t(k_2)}$ denotes resolution V – fractional factorial design replicate of 2^{k_1} and 2^{k_2} in ± 1 levels (cf. Narasimham *et al.*) [3]. The design points achieved from the transpose of incidence matrix of the design D_1 by multiplication are denoted by $[1-(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1)]2^{t(k_1)}$. Let $[1-(v,b_1,r_1,k_{11},k_{12}b_{11},b_{12},\lambda_1)]2^{t(k_1)}$ are the $b_1\times 2^{t(k_1)}$ design points achieved from D_1 by multiplication. $[a-(v,b_2,r_2,k_{21},k_{22},b_{21},b_{22},\lambda_2)]2^{t(k_2)}$ are the $b_2\times 2^{t(k_2)}$ design points achieved from D_2 by multiplication in Das and Narasimham [2] sense. Consider n_0 denote the number of central points.

The proposed method of construction of SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes is given below. Considering the method of construction of SORD using a pair of SUBA with two unequal block sizes having $b_1\times 2^{t(k_1)}+b_2\times 2^{t(k_2)}$ (cf. Narasimham *et al.* [3]) non-central design points (n). The set of 'n' non-central design points are extended to $2n$ design points (N) by adding 'n' $(n_0=n)$ central points $(0,0,....,0)$ just below or above the non-central design points. Hence, $2n(=N)$ be the total number of design points of the SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes.

Theorem

The design points $[1-(v,b_1,r_1,k_{11},k_{12},b_{11},b_{12},\lambda_1)]\times 2^{t(k_1)}U[(a-(v,b_2,r_2,k_{21},k_{22},b_{21},b_{22},\lambda_2)]\times 2^{t(k_2)}Un_0$ will give a v-dimensional SORD under tri-diagonal correlated error structure using a pair of SUBA with two unequal block sizes with $N=b_1\times 2^{t(k_1)}+b_2\times 2^{t(k_2)}Un_0$ design points, with $a^4=-\frac{(r_1-3\lambda_1)}{(r_2-3\lambda_2)}\times 2^{t(k_1)-t(k_2)}$.

Proof: For the design points generated from a pair of SUBA with two unequal block sizes, the simple symmetry conditions (2) are true. Further the conditions (3), (4) and (5) are true as follows:

$$\sum_{u=1}^{2n} x_{iu}^2=r_1 2^{t(k_1)}+r_2 2^{t(k_2)}a^2=\text{constant}=2n\gamma_2 \quad (11)$$

$$\sum_{u=1}^{2n} x_{iu}^4=r_1 2^{t(k_1)}+r_2 2^{t(k_2)}a^4=\text{constant}=3(2n)\gamma_4 \quad (12)$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2=\lambda_1 2^{t(k_1)}+\lambda_2 2^{t(k_2)}a^4=\text{constant}=2n\gamma_4 \quad (13)$$

From (12) and (13) we have,

$$r_1 2^{t(k_1)}+r_2 2^{t(k_2)}a^4=3\ (\lambda_1 2^{t(k_1)}+\lambda_2 2^{t(k_2)}a^4) \quad (14)$$

(i.e.,) $(r_1-3\lambda_1)2^{t(k_1)}+(r_2-3\lambda_2)2^{t(k_2)}a^4=0$ (15)

If $r_1=3\lambda_1$ and $r_2=3\lambda_2$, above equation (15) is satisfied. But to get a non- trivial design, at least one of levels '1' or 'a' must be different from 0 (zero). If $r_1\neq 3\lambda_1$ and $r_2\neq 3\lambda_2$, above equation(15) gives, $a^4=-\frac{(r_1-3\lambda_1)}{(r_2-3\lambda_2)}\times 2^{t(k_1)-t(k_2)}$. (Equation (16) has a real solution if either $r_1<3\lambda_1$ or $r_2<3\lambda_2$, but not both If the non-singularity condition (8) exists then only the design exists.

Example: A study on second order rotatable designs under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes can explained using v=6 factor with $D_1=(v=6,b_1=11,r_1=7,k_{11}=3,k_{12}=4,b_{11}=2,b_{12}=9,\lambda_1=4)$ and $D_2=(v=6,b_2=11,r_2=4,k_{21}=3,k_{22}=2,b_{21}=2,b_{22}=9,\lambda_2=1)$.

The design points, $[1-(v=6,b_1=11,r_1=7,k_{11}=3,k_{12}=4,b_{11}=2,b_{12}=9,\lambda_1=4)]2^{t(4)}U$ $[a-(v=6,b_2=11,r_2=4,k_{21}=3,k_{22}=2,b_{21}=2,b_{22}=9,\lambda_2=1)]2^3U(n_0=264)$ will give a SORD under tri-diagonal correlated structure of errors in N=528 design points, from (3), (4) and (5), we have,

$$\sum_{u=1}^{2n} x_{iu}^2=112+32a^2=2n\gamma_2 \quad (17)$$

$$\sum_{u=1}^{2n} x_{iu}^4=112+32a^4=3(2n)\gamma_4 \quad (18)$$

$$\sum_{u=1}^{2n} x_{iu}^2 x_{ju}^2=64+8a^4=2n\gamma_4 \quad (19)$$

On simplification of (18) and (19), we get a=1.074569. By substituting value 'a' in (17) and (18) we obtain $\gamma_2=0.403774$ and $\gamma_4=0.272727$. Non-singularity condition (8) is also satisfied. From (10) we have,

$$V(\hat{Y}_u)=\frac{2.181816(1+\rho)\sigma^2}{528[2.181816-0.978200(1-\rho)]}+[\frac{\sigma^2(1-\rho^2)}{213.192672}+2(\frac{-(0.403774)\sigma^2(1-\rho^2)}{528[2.181816-0.978200(1-\rho)]}]d^2$$
$$+\frac{\sigma^2(1-\rho^2)[1.909089-0.815167(1-\rho)]}{287.9997[2.181816-0.978200(1-\rho)]}d^4 \quad (20)$$

The variance function of SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes for factors $6\leq v\leq 15$ are given appendix in table 1

4. A study of dependence of the variance function of the response at different design points

Here, we study the dependence of variance function of response at different design points for SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes. Given 'v' factors different values of tri-diagonal correlated coefficient ρ and distance from center 'd' (centre of the design) between 0 and 1, the variances are tabulated. From (20) the variance of the estimated response is obtained by $V(\hat{Y}_u)=0.0035277$ (by taking $\rho=0.1$, $d=0.1$ and $\sigma=1$) For a given v, the study of variance function of response at different design points for SORD under tri-diagonal correlated structure of errors using pair of SUBA with two unequal block sizes for $6\leq v\leq 15$ and distance from center d for $d=0.1(0.1)1.0$ are tabulated. The numerical calculations are given in table 2.

The graphical representation for SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes for 6 factors in given in the appendix.

5. Conclusions

1. A new unified method of construction of second order rotatable designs under tri-diagonal correlated structure of errors using two suitably chosen SUBA with two unequal block sizes is suggested. From Table 1 and Table 2 of Appendix, we observed that
2. $V(\hat{Y}_u)$ is slightly increasing for different values of ρ is increasing.
3. $V(\hat{Y}_u)$ is slightly increasing for taking $d=0.1(0.1)1.0$, for given v, ρ.

Table 1: The variance function of SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes at different design points for factors $6 \le v \le 15$.

ρ	$D_1=(v=6,b_1=11,r_1=7,k_{11}=3,k_{12}=4,b_{11}=2,b_{12}=9,\lambda_1=4)$ $D_2=(v=6,b_2=11,r_2=4,k_{21}=3,k_{22}=2,b_{21}=2,b_{22}=9,\lambda_2=1)$ **N=528**	$D_1=(v=9,b_1=15,r_1=7,k_{11}=3,k_{12}=5,b_{11}=6,b_{12}=9,\lambda_1=3)$ $D_2=(v=9,b_2=30,r_2=9,k_{21}=2,k_{22}=3,b_{21}=9,b_{22}=21,\lambda_2=2)$ **N=960**
-0.9	$0.001278\sigma^2-0.000006\sigma^2d^2+0.000735\sigma^2d^4$	$0.000941\sigma^2+0.000058\sigma^2d^2+0.001468\sigma^2d^4$
-0.8	$0.001962\sigma^2+0.000014\sigma^2d^2+0.001311\sigma^2d^4$	$0.001323\sigma^2+0.000625\sigma^2d^2+0.002640\sigma^2d^4$
-0.7	$0.002389\sigma^2+0.000888\sigma^2d^2+0.001785\sigma^2d^4$	$0.001529\sigma^2+0.001283\sigma^2d^2+0.003632\sigma^2d^4$
-0.6	$0.002680\sigma^2+0.001414\sigma^2d^2+0.002179\sigma^2d^4$	$0.001659\sigma^2+0.001920\sigma^2d^2+0.004473\sigma^2d^4$
-0.5	$0.002891\sigma^2+0.001912\sigma^2d^2+0.002501\sigma^2d^4$	$0.001748\sigma^2+0.002501\sigma^2d^2+0.005174\sigma^2d^4$
-0.4	$0.003052\sigma^2+0.002360\sigma^2d^2+0.002756\sigma^2d^4$	$0.001813\sigma^2+0.003006\sigma^2d^2+0.005739\sigma^2d^4$
-0.3	$0.003178\sigma^2+0.002739\sigma^2d^2+0.002948\sigma^2d^4$	$0.001862\sigma^2+0.003424\sigma^2d^2+0.006172\sigma^2d^4$
-0.2	$0.003279\sigma^2+0.003046\sigma^2d^2+0.003078\sigma^2d^4$	$0.001901\sigma^2+0.003751\sigma^2d^2+0.006473\sigma^2d^4$
-0.1	$0.003363\sigma^2+0.003274\sigma^2d^2+0.003147\sigma^2d^4$	$0.001932\sigma^2+0.003985\sigma^2d^2+0.006643\sigma^2d^4$
0	$0.003433\sigma^2+0.003419\sigma^2d^2+0.003152\sigma^2d^4$	$0.001958\sigma^2+0.004122\sigma^2d^2+0.006684\sigma^2d^4$
0.1	$0.003492\sigma^2+0.003480\sigma^2d^2+0.003104\sigma^2d^4$	$0.001979\sigma^2+0.004163\sigma^2d^2+0.006595\sigma^2d^4$
0.2	$0.003543\sigma^2+0.003453\sigma^2d^2+0.002994\sigma^2d^4$	$0.001998\sigma^2+0.004101\sigma^2d^2+0.006377\sigma^2d^4$
0.3	$0.003588\sigma^2+0.003338\sigma^2d^2+0.002824\sigma^2d^4$	$0.002014\sigma^2+0.003941\sigma^2d^2+0.006030\sigma^2d^4$
0.4	$0.003627\sigma^2+0.003134\sigma^2d^2+0.002596\sigma^2d^4$	$0.002027\sigma^2+0.003682\sigma^2d^2+0.005554\sigma^2d^4$
0.5	$0.003661\sigma^2+0.002840\sigma^2d^2+0.002310\sigma^2d^4$	$0.002039\sigma^2+0.003323\sigma^2d^2+0.004950\sigma^2d^4$
0.6	$0.003692\sigma^2+0.002455\sigma^2d^2+0.001964\sigma^2d^4$	$0.002050\sigma^2+0.002861\sigma^2d^2+0.004217\sigma^2d^4$
0.7	$0.003720\sigma^2+0.001979\sigma^2d^2+0.001560\sigma^2d^4$	$0.002060\sigma^2+0.002297\sigma^2d^2+0.003355\sigma^2d^4$
0.8	$0.003744\sigma^2+0.001411\sigma^2d^2+0.001098\sigma^2d^4$	$0.002068\sigma^2+0.001634\sigma^2d^2+0.002365\sigma^2d^4$
0.9	$0.003767\sigma^2+0.000751\sigma^2d^2+0.000578\sigma^2d^4$	$0.002076\sigma^2+0.000866\sigma^2d^2+0.001246\sigma^2d^4$

ρ	$D_1=(v=10,b_1=15,r_1=8,k_{11}=4,k_{12}=6,b_{11}=5,b_{12}=10,\lambda_1=4)$ $D_2=(v=10,b_2=25,r_2=8,k_{21}=4,k_{22}=3,b_{21}=5,b_{22}=20,\lambda_2=2)$ N=1760
-0.9	$0.000721\sigma^2-0.000085\sigma^2d^2+0.000412\sigma^2d^4$
-0.8	$0.000892\sigma^2+0.000169\sigma^2d^2+0.000724\sigma^2d^4$
-0.7	$0.000568\sigma^2+0.000448\sigma^2d^2+0.000990\sigma^2d^4$
-0.6	$0.001013\sigma^2+0.000710\sigma^2d^2+0.001217\sigma^2d^4$
-0.5	$0.001041\sigma^2+0.000944\sigma^2d^2+0.001407\sigma^2d^4$
-0.4	$0.001061\sigma^2+0.001146\sigma^2d^2+0.001561\sigma^2d^4$
-0.3	$0.001075\sigma^2+0.001311\sigma^2d^2+0.001679\sigma^2d^4$
-0.2	$0.001086\sigma^2+0.001441\sigma^2d^2+0.001762\sigma^2d^4$
-0.1	$0.001095\sigma^2+0.001533\sigma^2d^2+0.001809\sigma^2d^4$
0	$0.001102\sigma^2+0.001587\sigma^2d^2+0.001821\sigma^2d^4$
0.1	$0.001108\sigma^2+0.001601\sigma^2d^2+0.001797\sigma^2d^4$
0.2	$0.001113\sigma^2+0.001579\sigma^2d^2+0.001738\sigma^2d^4$
0.3	$0.001118\sigma^2+0.001517\sigma^2d^2+0.001644\sigma^2d^4$
0.4	$0.001121\sigma^2+0.001416\sigma^2d^2+0.001515\sigma^2d^4$
0.5	$0.001125\sigma^2+0.001276\sigma^2d^2+0.001350\sigma^2d^4$
0.6	$0.001127\sigma^2+0.001099\sigma^2d^2+0.001150\sigma^2d^4$
0.7	$0.001130\sigma^2+0.000883\sigma^2d^2+0.000915\sigma^2d^4$
0.8	$0.001132\sigma^2+0.000627\sigma^2d^2+0.000645\sigma^2d^4$
0.9	$0.001134\sigma^2+0.000333\sigma^2d^2+0.000340\sigma^2d^4$

ρ	$D_1=(v=12,b_1=15,r_1=7,k_{11}=4,k_{12}=6,b_{11}=3,b_{12}=12,\lambda_1=3)$ $D_2=(v=12,b_2=13,r_2=4,k_{21}=3,k_{22}=4,b_{21}=4,b_{22}=9,\lambda_2=1)$ N=1376	$D_1=(v=15,b_1=16,r_1=8,k_{11}=5,k_{12}=9,b_{11}=6,b_{12}=10,\lambda_1=4)$ $D_2=(v=15,b_2=20,r_2=5,k_{21}=3,k_{22}=4,b_{21}=5,b_{22}=15,\lambda_2=1)$ N=4736
-0.9	$0.000870\sigma^2+0.000021\sigma^2d^2+0.000628\sigma^2d^4$	$0.001102\sigma^2-0.001587\sigma^2d^2+0.001821\sigma^2d^4$
-0.8	$0.001103\sigma^2+0.000398\sigma^2d^2+0.001132\sigma^2d^4$	$0.001108\sigma^2+0.001601\sigma^2d^2+0.001797\sigma^2d^4$
-0.7	$0.001211\sigma^2+0.000801\sigma^2d^2+0.001566\sigma^2d^4$	$0.001113\sigma^2+0.001579\sigma^2d^2+0.001738\sigma^2d^4$
-0.6	$0.001273\sigma^2+0.001178\sigma^2d^2+0.001937\sigma^2d^4$	$0.001118\sigma^2+0.001517\sigma^2d^2+0.001644\sigma^2d^4$
-0.5	$0.001314\sigma^2+0.001511\sigma^2d^2+0.002249\sigma^2d^4$	$0.001121\sigma^2+0.001416\sigma^2d^2+0.001515\sigma^2d^4$
-0.4	$0.001343\sigma^2+0.001795\sigma^2d^2+0.002502\sigma^2d^4$	$0.001125\sigma^2+0.001276\sigma^2d^2+0.001350\sigma^2d^4$
-0.3	$0.001364\sigma^2+0.002028\sigma^2d^2+0.002697\sigma^2d^4$	$0.001127\sigma^2+0.001099\sigma^2d^2+0.001150\sigma^2d^4$
-0.2	$0.001380\sigma^2+0.002207\sigma^2d^2+0.002834\sigma^2d^4$	$0.001130\sigma^2+0.000883\sigma^2d^2+0.000915\sigma^2d^4$
-0.1	$0.001393\sigma^2+0.002330\sigma^2d^2+0.002914\sigma^2d^4$	$0.001132\sigma^2+0.000627\sigma^2d^2+0.000645\sigma^2d^4$
0	$0.001404\sigma^2+0.002399\sigma^2d^2+0.002936\sigma^2d^4$	$0.001134\sigma^2+0.000333\sigma^2d^2+0.000340\sigma^2d^4$
0.1	$0.001412\sigma^2+0.002412\sigma^2d^2+0.002901\sigma^2d^4$	$0.001512\sigma^2+0.003412\sigma^2d^2+0.002501\sigma^2d^4$
0.2	$0.001420\sigma^2+0.002370\sigma^2d^2+0.002808\sigma^2d^4$	$0.001520\sigma^2+0.003370\sigma^2d^2+0.002608\sigma^2d^4$
0.3	$0.001426\sigma^2+0.002271\sigma^2d^2+0.002657\sigma^2d^4$	$0.001526\sigma^2+0.003271\sigma^2d^2+0.002557\sigma^2d^4$
0.4	$0.001431\sigma^2+0.002116\sigma^2d^2+0.002450\sigma^2d^4$	$0.001531\sigma^2+0.003116\sigma^2d^2+0.002350\sigma^2d^4$

0.5	$0.001436\sigma^2+0.001905\sigma^2d^2+0.002185\sigma^2d^4$	$0.001536\sigma^2+0.003905\sigma^2d^2+0.002285\sigma^2d^4$
0.6	$0.001440\sigma^2+0.001637\sigma^2d^2+0.001862\sigma^2d^4$	$0.001540\sigma^2+0.003637\sigma^2d^2+0.001262\sigma^2d^4$
0.7	$0.001444\sigma^2+0.001312\sigma^2d^2+0.001483\sigma^2d^4$	$0.001544\sigma^2+0.003312\sigma^2d^2+0.001383\sigma^2d^4$
0.8	$0.001447\sigma^2+0.000931\sigma^2d^2+0.001046\sigma^2d^4$	$0.001547\sigma^2+0.003931\sigma^2d^2+0.001546\sigma^2d^4$
0.9	$0.001450\sigma^2+0.000494\sigma^2d^2+0.000551\sigma^2d^4$	$0.001550\sigma^2+0.003494\sigma^2d^2+0.000651\sigma^2d^4$

Table 2: Study of dependence of estimated SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes at different design points for $6 \le v \le 15$ for different values of ρ, d and $\sigma = 1$.

$$D_1 = (v=6, b_1=11, r_1=7, k_{11}=3, k_{12}=4, b_{11}=2, b_{12}=9, \lambda_1=4)$$
$$D_2 = (v=6, b_2=11, r_2=4, k_{21}=3, k_{22}=2, b_{21}=2, b_{22}=9, \lambda_2=1)$$ **n=264, N=528**

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
-0.9	0.00192	0.00195	0.00278	0.00280	0.00291	0.00312	0.00328	0.00415	0.00415	0.00419
-0.8	0.00212	0.00358	0.00458	0.00459	0.00459	0.00500	0.00533	0.00552	0.00617	0.00620
-0.7	0.00215	0.00292	0.00311	0.00342	0.00355	0.00365	0.00433	0.00441	0.00448	0.00522
-0.6	0.00225	0.00600	0.00487	0.00503	0.00349	0.00438	0.00331	0.00419	0.00512	0.00578
-0.5	0.00389	0.00711	0.00941	0.00542	0.00401	0.00507	0.00335	0.00417	0.00555	0.00612
-0.4	0.00422	0.00794	0.00511	0.00570	0.00443	0.00564	0.00324	0.00400	0.00574	0.00628
-0.3	0.00422	0.00848	0.00138	0.00593	0.00477	0.00608	0.00313	0.00382	0.00589	0.00667
-0.2	0.00518	0.00912	0.00212	0.00612	0.00502	0.00642	0.00296	0.00355	0.00589	0.00667
-0.1	0.00513	0.00915	0.00246	0.00626	0.00519	0.00663	0.00203	0.00321	0.00456	0.00785
0	0.00613	0.00945	0.00274	0.00637	0.00527	0.00673	0.00254	0.00287	0.00465	0.00774
0.1	0.00713	0.00949	0.00273	0.00644	0.00526	0.00672	0.00227	0.00244	0.00272	0.00733
0.2	0.00714	0.00937	0.00236	0.00647	0.00518	0.00659	0.00328	0.00443	0.00271	0.00731
0.3	0.00714	0.00904	0.00198	0.00646	0.00500	0.00635	0.00325	0.00427	0.00277	0.00835
0.4	0.00814	0.00847	0.00112	0.00642	0.00475	0.00599	0.00317	0.00405	0.00271	0.00824
0.5	0.00816	0.00745	0.00031	0.00634	0.00441	0.00552	0.00305	0.00375	0.00265	0.00813
0.6	0.00816	0.00778	0.00458	0.00623	0.00398	0.00493	0.00289	0.00336	0.00255	0.00896
0.7	0.00716	0.00785	0.00712	0.00607	0.00347	0.00423	0.00268	0.00292	0.00243	0.00803
0.8	0.00718	0.00790	0.00607	0.00588	0.00287	0.00341	0.00244	0.00240	0.00231	0.00554
0.9	0.00718	0.00790	0.00419	0.00520	0.00525	0.00558	0.00614	0.00580	0.00615	0.00627

$$D_1 = (v=9, b_1=15, r_1=7, k_{11}=3, k_{12}=5, b_{11}=6, b_{12}=9, \lambda_1=3)$$
$$D_2 = (v=9, b_2=30, r_2=9, k_{21}=2, k_{22}=3, b_{21}=9, b_{22}=21, \lambda_2=2)$$ **n=480, N=960**

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
-0.9	0.00094	0.00094	0.00095	0.00098	0.00104	0.00115	0.00131	0.00157	0.00194	0.00245
-0.8	0.00132	0.00135	0.00140	0.00149	0.00164	0.00189	0.00226	0.00280	0.00355	0.00458
-0.7	0.00154	0.00158	0.00167	0.00182	0.00206	0.00245	0.00301	0.00382	0.00494	0.00644
-0.6	0.00167	0.00174	0.00186	0.00207	0.00241	0.00291	0.00365	0.00471	0.00609	0.00805
-0.5	0.00177	0.00184	0.00200	0.00228	0.00269	0.00331	0.00420	0.00546	0.00716	0.00941
-0.4	0.00184	0.00194	0.00212	0.00243	0.00291	0.00363	0.00465	0.00606	0.00798	0.01051
-0.3	0.00189	0.00194	0.00215	0.00250	0.00309	0.00387	0.00499	0.00653	0.00861	0.01138
-0.2	0.00193	0.00205	0.00228	0.00265	0.00322	0.00406	0.00524	0.00688	0.00909	0.01200
-0.1	0.00193	0.00206	0.00230	0.00269	0.00328	0.00415	0.00539	0.00709	0.00938	0.01246
0	0.00194	0.00207	0.00232	0.00272	0.00333	0.00422	0.00548	0.00722	0.00955	0.01274
0.1	0.00194	0.00207	0.00232	0.00272	0.00333	0.00421	0.00553	0.00718	0.00948	0.01273
0.2	0.00194	0.00207	0.00232	0.00271	0.00331	0.00419	0.00542	0.00780	0.00935	0.01236
0.3	0.00203	0.00216	0.00239	0.00277	0.00335	0.00417	0.00535	0.00694	0.00908	0.01198
0.4	0.00203	0.00215	0.00236	0.00271	0.00324	0.00400	0.00508	0.00655	0.00851	0.01112
0.5	0.00203	0.00213	0.00233	0.00265	0.00313	0.00382	0.00479	0.00611	0.00788	0.01031
0.6	0.00202	0.00211	0.00228	0.00255	0.00296	0.00355	0.00438	0.00551	0.00702	0.00912
0.7	0.00202	0.00209	0.00222	0.00243	0.00203	0.00321	0.00387	0.00475	0.00594	0.00771
0.8	0.00201	0.00206	0.00216	0.00231	0.00254	0.00287	0.00333	0.00396	0.00480	0.00606
0.9	0.00200	0.00203	0.00208	0.00215	0.00227	0.00244	0.00268	0.00300	0.00351	0.00418

$$D_1 = (v=10, b_1=15, r_1=8, k_{11}=4, k_{12}=6, b_{11}=5, b_{12}=10, \lambda_1=4)$$
$$D_2 = (v=10, b_2=25, r_2=8, k_{21}=4, k_{22}=3, b_{21}=5, b_{22}=20, \lambda_2=2)$$ n=880, N=1760

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
-0.9	0.00072	0.00071	0.00071	0.00071	0.00072	0.00074	0.00077	0.00083	0.00092	0.00104
-0.8	0.00089	0.00089	0.00091	0.00093	0.00097	0.00104	0.00114	0.00129	0.00150	0.00178
-0.7	0.00057	0.00058	0.00061	0.00066	0.00074	0.00085	0.00102	0.00125	0.00157	0.00200
-0.6	0.00102	0.00104	0.00108	0.00115	0.00126	0.00142	0.00165	0.00196	0.00238	0.00294
-0.5	0.00105	0.00108	0.00113	0.00122	0.00136	0.00156	0.00184	0.00222	0.00272	0.00339
-0.4	0.00107	0.00110	0.00117	0.00128	0.00144	0.00167	0.00199	0.00243	0.00301	0.00376

-0.3	0.00108	0.00113	0.00120	0.00132	0.00150	0.00176	0.00212	0.00260	0.00323	0.00406
-0.2	0.00110	0.00114	0.00122	0.00136	0.00155	0.00183	0.00221	0.00272	0.00340	0.00428
-0.1	0.00111	0.00115	0.00124	0.00138	0.00159	0.00188	0.00228	0.00281	0.00352	0.00443
0	0.00111	0.00116	0.00125	0.00140	0.00161	0.00190	0.00231	0.00286	0.00358	0.00451
0.1	0.00112	0.00117	0.00126	0.00141	0.00162	0.00191	0.00232	0.00286	0.00358	0.00450
0.2	0.00112	0.00117	0.00126	0.00140	0.00161	0.00190	0.00230	0.00283	0.00353	0.00443
0.3	0.00113	0.00118	0.00126	0.00140	0.00159	0.00187	0.00225	0.00276	0.00342	0.00427
0.4	0.00113	0.00118	0.00126	0.00138	0.00156	0.00182	0.00217	0.00264	0.00326	0.00405
0.5	0.00113	0.00117	0.00124	0.00136	0.00152	0.00175	0.00207	0.00249	0.00304	0.00375
0.6	0.00113	0.00117	0.00123	0.00133	0.00147	0.00167	0.00194	0.00230	0.00277	0.00336
0.7	0.00113	0.00116	0.00121	0.00129	0.00140	0.00156	0.00178	0.00206	0.00244	0.00292
0.8	0.00113	0.00115	0.00119	0.00124	0.00132	0.00144	0.00159	0.00179	0.00206	0.00240
0.9	0.00113	0.00114	0.00116	0.00119	0.00123	0.00129	0.00137	0.00148	0.00162	0.00180

$$D_1 = (v = 12, b_1 = 15, r_1 = 7, k_{11} = 4, k_{12} = 6, b_{11} = 3, b_{12} = 12, \lambda_1 = 3)$$

$$D_2 = (v = 12, b_2 = 13, r_2 = 4, k_{21} = 3, k_{22} = 4, b_{21} = 4, b_{22} = 9, \lambda_2 = 1)$$ **n=688, N=1376**

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
-0.9	0.00087	0.00087	0.00087	0.00088	0.00091	0.00095	0.00103	0.00114	0.00129	0.00151
-0.8	0.00110	0.00112	0.00114	0.00119	0.00127	0.00139	0.00156	0.00182	0.00216	0.00263
-0.7	0.00121	0.00124	0.00129	0.00137	0.00150	0.00170	0.00197	0.00236	0.00288	0.00357
-0.6	0.00128	0.00132	0.00139	0.00151	0.00168	0.00194	0.00231	0.00281	0.00349	0.00438
-0.5	0.00132	0.00137	0.00146	0.00161	0.00183	0.00214	0.00259	0.00320	0.00401	0.00507
-0.4	0.00136	0.00141	0.00152	0.00169	0.00194	0.00231	0.00282	0.00351	0.00443	0.00564
-0.3	0.00138	0.00144	0.00156	0.00175	0.00203	0.00244	0.00300	0.00376	0.00477	0.00608
-0.2	0.00140	0.00147	0.00160	0.00180	0.00210	0.00254	0.00314	0.00395	0.00502	0.00642
-0.1	0.00141	0.00149	0.00162	0.00183	0.00215	0.00260	0.00323	0.00407	0.00519	0.00663
0	0.00142	0.00150	0.00164	0.00186	0.00218	0.00264	0.00328	0.00414	0.00527	0.00673
0.1	0.00143	0.00151	0.00165	0.00187	0.00219	0.00265	0.00328	0.00414	0.00526	0.00672
0.2	0.00144	0.00151	0.00165	0.00187	0.00218	0.00263	0.00325	0.00408	0.00518	0.00659
0.3	0.00144	0.00152	0.00165	0.00185	0.00215	0.00258	0.00317	0.00396	0.00500	0.00635
0.4	0.00145	0.00151	0.00164	0.00183	0.00211	0.00250	0.00305	0.00378	0.00475	0.00599
0.5	0.00145	0.00151	0.00162	0.00179	0.00204	0.00240	0.00289	0.00354	0.00441	0.00552
0.6	0.00145	0.00150	0.00160	0.00174	0.00196	0.00227	0.00268	0.00324	0.00398	0.00493
0.7	0.00145	0.00149	0.00157	0.00169	0.00186	0.00210	0.00244	0.00289	0.00347	0.00423
0.8	0.00145	0.00148	0.00153	0.00161	0.00174	0.00190	0.00214	0.00246	0.00287	0.00341
0.9	0.00145	0.00147	0.00149	0.00154	0.00160	0.00169	0.00182	0.00198	0.00220	0.00249

$$D_1 = (v = 15, b_1 = 16, r_1 = 8, k_{11} = 5, k_{12} = 9, b_{11} = 6, b_{12} = 10, \lambda_1 = 4)$$

$$D_2 = (v = 15, b_2 = 20, r_2 = 5, k_{21} = 3, k_{22} = 4, b_{21} = 5, b_{22} = 15, \lambda_2 = 1)$$ n=2368, N=4736

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
-0.9	0.00089	0.00089	0.00088	0.00087	0.00087	0.00088	0.00091	0.00097	0.00106	0.00119
-0.8	0.00092	0.00092	0.00094	0.00096	0.00101	0.00108	0.00119	0.00136	0.00160	0.00193
-0.7	0.00093	0.00095	0.00098	0.00103	0.00111	0.00124	0.00144	0.00171	0.00208	0.00258
-0.6	0.00094	0.00096	0.00101	0.00109	0.00121	0.00139	0.00165	0.00201	0.00249	0.00314
-0.5	0.00094	0.00098	0.00104	0.00113	0.00129	0.00151	0.00183	0.00227	0.00285	0.00363
-0.4	0.00095	0.00099	0.00106	0.00117	0.00135	0.00161	0.00197	0.00247	0.00314	0.00400
-0.3	0.00095	0.00099	0.00108	0.00121	0.00140	0.00169	0.00209	0.00264	0.00338	0.00434
-0.2	0.00095	0.00100	0.00109	0.00123	0.00144	0.00175	0.00218	0.00277	0.00355	0.00458
-0.1	0.00095	0.00100	0.00110	0.00125	0.00147	0.00179	0.00224	0.00285	0.00366	0.00473
0	0.00096	0.00101	0.00110	0.00126	0.00148	0.00181	0.00227	0.00289	0.00372	0.00479
0.1	0.00096	0.00101	0.00111	0.00126	0.00149	0.00182	0.00227	0.00289	0.00371	0.00478
0.2	0.00096	0.00101	0.00110	0.00125	0.00148	0.00180	0.00224	0.00284	0.00364	0.00468
0.3	0.00096	0.00101	0.00110	0.00124	0.00145	0.00176	0.00218	0.00276	0.00351	0.00450
0.4	0.00095	0.00100	0.00108	0.00122	0.00142	0.00170	0.00209	0.00263	0.00333	0.00424
0.5	0.00095	0.00100	0.00107	0.00119	0.00137	0.00162	0.00198	0.00245	0.00308	0.00390
0.6	0.00095	0.00099	0.00105	0.00116	0.00131	0.00153	0.00182	0.00223	0.00277	0.00347
0.7	0.00095	0.00098	0.00103	0.00111	0.00124	0.00141	0.00165	0.00198	0.00241	0.00297
0.8	0.00095	0.00097	0.00109	0.00106	0.00115	0.00127	0.00145	0.00168	0.00198	0.00237
0.9	0.00094	0.00096	0.00097	0.00101	0.00105	0.00112	0.00121	0.00133	0.00149	0.00170

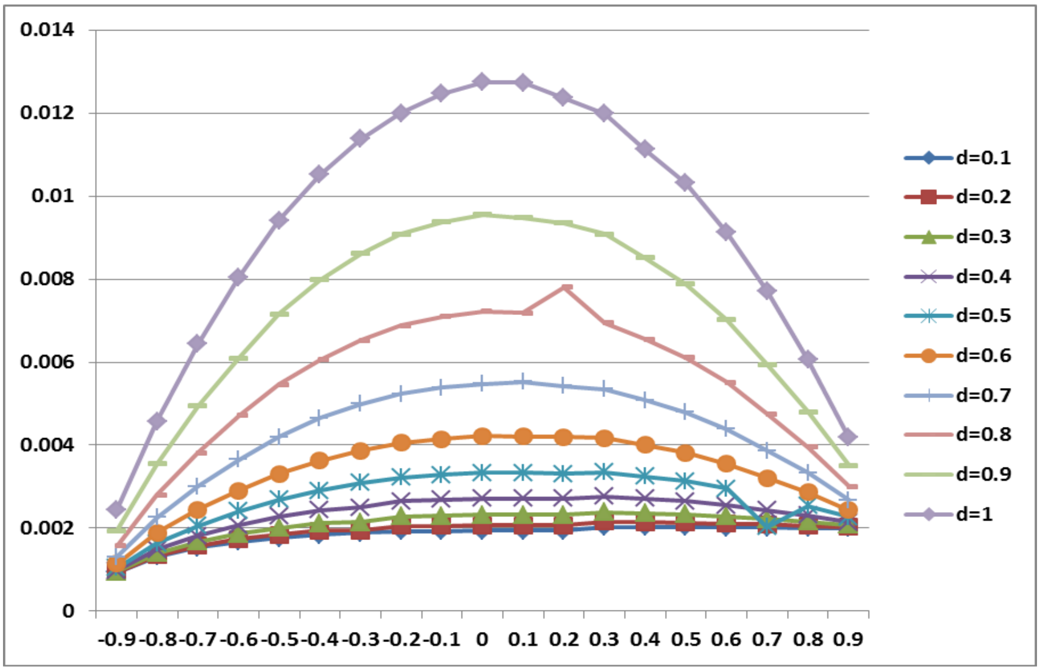

Fig 1: Graphical representation for SORD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes with parameters $D_1 = (v = 6, b_1 = 11, r_1 = 7, k_{11} = 3, k_{12} = 4, b_{11} = 2, b_{12} = 9, \lambda_1 = 4)$ and $D_2 = (v = 6, b_2 = 11, r_2 = 4, k_{21} = 3, k_{22} = 2, b_{21} = 2, b_{22} = 9, \lambda_2 = 1)$ for 6 factors with N=528.

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