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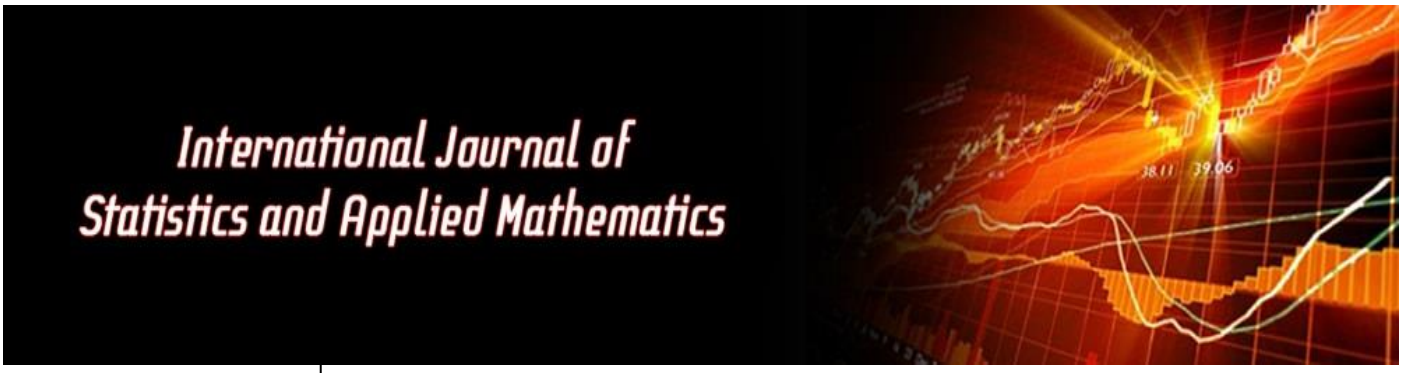

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Use of the theory of Euclidean distance in testing for multivariate normality with application to breast cancer diagnostic data

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Abstract
This paper presents an adaptive technique for assessing multivariate normality (*MVN*). It is shown that the squared L_2 norm otherwise known as the squared Euclidean distance of a standard *d*-variate normal distribution is chi-squared distributed with *d* degrees of freedom. Based on this, an adaptive test for *MVN* was proposed as the sum of squared differences between the ordered set of the squared normalized L_2 norms of the observation vectors and the set of the population *p*th quantiles from the chi-squared distribution with *d* degrees of freedom. The critical values of the test were evaluated for different sample sizes and different number of random variables contained in the multivariate data through extensive simulations. For some selected sample sizes and number of random variables, the empirical power of the proposed test was compared with those of some other widely used techniques for assessing multivariate normality. The results showed that the test can be recommended as a good tool for testing *MVN* of a dataset especially for large sample cases. The test was applied to a data set extracted from the Wisconsin breast cancer diagnostic data and the result showed that the data set was not multivariate normal.

Keywords: Euclidean distance, multivariate normality, *p*th quantile; probability plots, empirical power of a test

1. Introduction

Suppose a *d*-component random vector $\mathbf{x} \in R^d$ is defined by a distribution function $F(\mathbf{x})$ with probability function $f(\mathbf{x})$. Let $F_0(\mathbf{x})$ be a distribution function of a multivariate normal population having mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, with probability density function $f_0(\mathbf{x})$. Suppose a sample of *n* independent and identically distributed (*iid*) observation vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ is available from an unknown continuous distribution function $F(\mathbf{x})$. Testing the assumption of multivariate normality (*MVN*) of the data set involves a statistical determination of whether or not the unknown distribution function $F(\mathbf{x})$ conforms to the known distribution function $F_0(\mathbf{x})$.

There is no shortage of procedures for obtaining goodness-of-fit test for *MVN*. This is made possible by the various characterizations of the multivariate normal distribution such as the characteristic function, the moment generating function, the Laplace transform as well as measures of entropy, skewness and kurtosis. In fact, several different statistics have been developed for assessing *MVN* by the use of each of these and many other characterizations. This therefore presents these procedures in classes according to each characterization. Some of the tests include Kuwana and Kariya [1]; Baxter and Gale [2]; Dufour, Khalaf and Beaulieu [3] and Koizumi, Hyodo and Pavlenko [4].

One class of tests for *MVN* of multivariate datasets that have received attention of researchers is the probability plots class of tests for *MVN*. Suppose $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ is a random sample of *n* observation vectors from a *d*-dimensional multivariate normal population having mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. Let $\bar{\mathbf{x}}$ and $\mathbf{S}$ be estimators of $\boldsymbol{\mu}$ and Σ respectively. It is well known that the squared radii

$$y_j = (\mathbf{x}_j - \bar{\mathbf{x}})^T \mathbf{S}^{-1} (\mathbf{x}_j - \bar{\mathbf{x}}); \quad j = 1, 2, ..., n \tag{1}$$

are asymptotically distributed as a chi-squared distribution with d degrees of freedom, where $\bar{\mathbf{x}} = n^{-1}\sum_{j=1}^{n}\mathbf{x}_j$ and $\mathbf{S} = (n-1)^{-1}\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})^T$

Healy [5] obtained a graphical plot of the ordered squared radii, $y_{(j)};\ j = 1, 2, \ldots, n$ versus the approximate expected order statistics from the chi-squared distribution with d degrees of freedom. As a means of assessing *MVN* of the data set, he stated that the *MVN* of the data set may be rejected if the plot fails to be approximately linear. Madukaife and Okafor [6] used this transformation to obtain a formal test for *MVN* based on probability plots. They ordered the squared radii to have the order statistics $y_{(j)}; j = 1, 2, \cdots, n$ and compared the jth order statistic with a corresponding jth approximate expected order statistic $E(y_{(j)})$. They obtained the jth approximate expected order statistic as the p_jth quantile of the chi squared distribution with d degrees of freedom. That is, $E(y_{(j)})$ is the inverse distribution function $F^{-1}(p_j)$ of the chi square distribution with $p_j = n^{-1}(j - 0.5)$. Precisely, their test statistic is the sum of squared differences between the observed and corresponding expected order statistics. They concluded that multivariate normality of a data set will be rejected for large values of the statistic.

It is important to note that the squared radii given in (1) are obtained by the principle of sample Mahalanobis squared distance of multivariate observations. On the assumption of multivariate normality, the sample Mahalanobis squared distance is not the only transform that gives rise to a set of chi-square observations with d degrees of freedom. This paper proposes to obtain another chi – squared transform by the principle of Euclidean distance which can be adapted to the test statistic according to Madukaife and Okafor [6]. The rest of the paper is organized as follows: the technique is proposed in the immediate next section while the empirical critical values of the proposed test follow after it as a section. Thereafter, the powers of the test are compared with the powers of other good omnibus tests for *MVN* in the literature. The proposed test is applied to a real-life data set and finally, the paper ends with the concluding section

2. The proposed test

Suppose $\mathbf{x} \in R^d \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The L_2 norm of $\mathbf{x}$, also known as the Euclidean distance or Euclidean norm of the vector, is defined by $\|\mathbf{x}\|_2 = (\mathbf{x}'\mathbf{x})^{1/2}$. The squared Euclidean distance $\|\mathbf{x}\|_2^2 = \mathbf{x}'\mathbf{x}$ is simply sum of squares of dependent normal random variables. Kettani and Ostrouchov [7] have obtained the distribution of this $\mathbf{x}'\mathbf{x}$ for different forms of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Precisely, they obtained the characteristic function of this univariate transform as:

$$Q(t) = \left|\mathbf{I}_d + 2it\boldsymbol{\Sigma}\right|^{-1/2} etr\left\{-it\left(\mathbf{I}_d + 2it\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\right\} \quad (2)$$

The characteristic function in (2) reduces to $Q(t) = (1 + 2it)^{-d/2}$ which is that of chi-squared distribution with d degrees of freedom when $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_d$ since the dependent normal random variables are transformed to independent standard normal variables. Our problem now is to have $\mathbf{x} \in R^d \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ transformed to $\mathbf{z} \in R^d \sim N_d(\mathbf{0}, \mathbf{I}_d)$. This can be achieved in different ways but one tractable means is the use of matrix diagonalization of $\boldsymbol{\Sigma}$ through orthogonal transformation of principal components.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a random sample of size n from $N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma}$ be estimated by $\bar{\mathbf{x}}$ and $\mathbf{s}$ respectively, then

$$y_{ij} = \mathbf{h}_i'\mathbf{x}_j;\ i = 1, 2, \ldots, d,\ j = 1, 2, \ldots, n \quad (3)$$

is the jth observation of the ith principal component of $\mathbf{x}$, where $\mathbf{h}_i$ is the normalized eigenvector associated with the ith estimated eigenvalue of the random vector $\mathbf{x}$. Hanusz and Tarasinska[8] standardized the y_{ij} observations in each ith principal component given in (3) in order to have observations that do not depend on the sample covariance matrix S and obtained

$$z_{ij} = \frac{(y_{ij} - \bar{Y}_i)\sqrt{n}}{\sqrt{(n+1)\lambda_i}};\ i = 1, 2, \ldots, d;\ j = 1, 2, \ldots, n \quad (4)$$

such that under the assumption of multinormality of $\mathbf{x}$, z_{ij} will be a jth observation of ith approximately independent standard normal principal component of $\mathbf{x}$. Now, let a random sample of n observation vectors be transformed to $z_{ij};\ i = 1, 2, \ldots, d;\ j = 1, 2, \ldots, n$ according to (4). If the parent distribution function of the sample is d-variate normal, then

$$w_j = \sum_{i=1}^{d} z_{ij}^2;\ j = 1, 2, \ldots, n$$

is the jth chi-squared observation with d degrees of freedom. The w_j's are ordered as $w_{(1)} \le w_{(2)} \le \ldots \le w_{(n)}$ with their corresponding approximate expected chi-squared order statistics $c_j;\ j = 1, 2, \ldots, n$ obtained for each j as the inverse distribution function $F^{-1}((j - 0.5)/n)$ from the chi-squared distribution with d degrees of freedom. A test for *MVN*, adapted from Madukaife and Okafor[6] is therefore proposed as the sum of squared differences between each observed chi-squared value and the corresponding approximate expected chi-squared value. That is, the proposed statistic is

$$L_{n,d}^2 = \sum_{i=1}^{d}\left(w_{(j)} - c_j\right)^2;\ j = 1, 2, \ldots, n \quad (5)$$

The test rejects multivariate normality of the data set for large value of the statistic.

3. Empirical critical values of the test

The empirical critical values of the test statistic for different combinations of the sample size n and the number of variables d through extensive simulation studies are obtained here. Precisely, the critical values at 0.5, 1, 2.5 and 5 percent levels of significance for n =10, 25, 50, 100 and 500 and d = 2, 3, 4 and 5 are evaluated. N = 100,000 samples were generated for $n \le 100$ and N = 50,000 samples for $n > 100$ from a d-dimensional multivariate standard normal distribution and the N values of the statistic were obtained from each generated set of N samples under each specified n and d. The α-level critical value of the test for the n and d is then obtained as the $100(1-\alpha)$ percentile of the N values. The percentile values are presented in Table 1.

Table 1: Empirical critical values of the $L^2_{n,d}$ statistic at α = 0.005, 0.01, 0.025 and 0.05

Sample size	*d* = 2 .				*d* = 3 .			
	0.005	0.01	0.025	0.05	0.005	0.01	0.025	0.05
10	15.7636	14.7111	13.0184	11.5694	27.0907	25.3376	22.8213	20.6882
25	27.9985	24.3414	20.0024	16.8647	36.6072	33.1163	28.4596	24.7529
50	49.8609	39.0805	28.5069	22.9887	54.4440	45.8927	36.8452	30.8073
100	71.6042	55.4426	38.7936	30.0632	78.3787	61.9425	46.8867	37.9664
500	101.1215	81.9565	60.7241	46.4818	117.3302	96.8911	70.1829	55.4076

Sample size	*d* = 4 .				*d* = 5 .			
	0.005	0.01	0.025	0.05	0.005	0.01	0.025	0.05
10	39.9518	37.8190	34.5390	31.8054	55.3487	52.8934	49.0957	45.7739
25	49.6909	45.3328	39.3123	34.4060	65.1890	59.6608	51.9946	45.8699
50	63.0315	55.6148	46.0307	39.1501	74.3686	66.6808	56.5207	48.3534
100	85.2929	71.5301	55.9270	45.8705	95.2191	80.7978	65.6625	54.8823
500	129.3847	105.4645	80.9276	64.5576	146.0929	117.3151	89.8026	72.8074

4. Empirical power studies

In this section, empirical power performance of the proposed test ($L^2_{n,d}$) is compared with the powers of some other omnibus tests for *MVN* most of which are based on the probability plots and the comparison is made through extensive simulation studies. The competing tests considered here include the Henze and Zirkler (*HZ*) test for *MVN*, Henze and Zirkler [9]; the Singh's classical (S_{cl}) test for MVN, Singh[10]; the Madukaife and Okafor ($T_{n,d}$, $G_{n,d}$) tests for multivariate normality, Madukaife and Okafor [11, 6]. The competing test procedures are described in what follows.

4.1 Description of the competing test statistics

Henze and Zirkler Test: Henze and Zirkler [9] obtained a class of affine invariant and consistent test for multivariate normality based on the distance between the empirical characteristic function and the theoretical characteristic function of the multivariate normal distribution. The statistic is of the form

$$T_{n,\beta} = n(4I(S \text{ is singular}) + D_{n,\beta} I(S \text{ is nonsingular})),$$

where

$$D_{n,\beta} = \frac{1}{n^2}\sum_{j,k=1}^{n} \exp\left\{-\frac{\beta^2}{2}\left\|\underline{y}_j - \underline{y}_k\right\|^2\right\}$$

$$-2\left(1+\beta^2\right)^{-d/2}\frac{1}{n}\sum_{j=1}^{n}\exp\left\{-\frac{\beta^2\left\|\underline{y}_j\right\|^2}{2\left(1+\beta^2\right)}\right\}+\left(1+2\beta^2\right)^{-d/2}$$

$y_j = S^{-1/2}(\mathbf{x}_j - \overline{X})$; *I(.)* is an indicator function and *S* is the sample covariance matrix of the multivariate data set. They recommended the smoothing parameter, β to be obtained from kernel density estimation by

$$\beta = \beta_d(n) = \frac{1}{\sqrt{2}}\left(\frac{2d+1}{4}\right)^{\frac{1}{d+4}} n^{\frac{1}{d+4}}$$

They tried to obtain the asymptotic null distribution of this test as a weighted chi-square at 1 degree of freedom. However, they obtained the empirical critical values of the test and applied the test based on the empirical critical values.

Singh's Classical Test: Singh [10] obtained a classical test for multivariate normality based on the product moment correlation coefficient between an ordered beta transform, $z_{(j)}; j = 1, 2, \ldots, n$ of a multivariate data set and the set of expected beta order statistics, $c_j; j = 1, 2, \ldots, n$. The beta of the first kind transform was obtained according to Gnanadesikan and Kettenring [12] by further transforming $y_j;\ j = 1, 2, \ldots, n$ in (1) as $z_j = n(n-1)^{-2} y_j;\ j = 1, 2, \ldots, n$. They concluded that multivariate normality of the data set will be rejected if the correlation coefficient is not close to 1. The correlation coefficient test statistic is given as:

$$r = \frac{\sum_{j=1}^{n}\left(z_{(j)} - \overline{z}\right)\left(c_{(j)} - \overline{c}\right)}{\sqrt{\sum_{j=1}^{n}\left(z_{(j)} - \overline{z}\right)^2}\sqrt{\sum_{j=1}^{n}\left(c_{(j)} - \overline{c}\right)^2}};$$

Where $\overline{z} = \sum_{j=1}^{n} z_{(j)}/n$ and $\overline{c} = \sum_{j=1}^{n} c_{(j)}/n$.

Also, the critical values for this test were obtained empirically via simulation studies.

Madukaife and Okafor $T_{n,d}$ and $G_{n,d}$ Tests: Madukaife and Okafor [11, 6] respectively obtained two tests for assessing the assumption of *MVN* of multivariate data sets. The test procedures have the statistics:

$$T_n = \sum_{i=1}^{d}\sum_{j=1}^{n}(z_{i(j)} - c_j)^2 \quad \text{and} \quad G_n = \sum_{j=1}^{n}(y_{(j)} - c_j)^2,$$

where $z_{i(j)}$ is the *j*th sample order statistic in the *i*th principal component obtained from the multivariate data set according to (4), $y_{(j)}$ is the *j*th sample order statistic of the chi squared transform observations according to (1). In each statistic, c_j is the *j*th expected order statistic. They obtained the critical values of the tests empirically via extensive simulation studies said that *MVN* of data sets are rejected for large values of the statistics.

4.2 Power Comparison of the Tests

In this section, 10,000 data sets were generated in each combination of *n* = 25, 50, 100 and 500 and *d* = 2 and 5 from

11 different multivariate distributions. The values of each of the competing statistics being compared were evaluated in each of the 10,000 simulated samples for each n and d. The empirical power of each test statistic was obtained as the percentage of the 10,000 samples that is rejected by the statistic at α of 5%. The distributions include the standard multivariate normal distribution (*MVN*); the multivariate Cauchy distribution (*MVC*) with identity covariance matrix; the multivariate t distribution (*MVt*) with identity covariance matrix and 2 degrees of freedom and multivariate generalized extreme value distribution (*Mvgevd*) with dependent parameter 0.72. Others are products of some univariate distributions such as student's t with 2 degree of freedom, standard lognormal, beta of the first kind with 1 and 5 parameters, standard exponential, standard arcsine and standard normal. The power performance is presented in Tables 2 and 3 respectively for $d = 2$ and 5.

From Tables 2 and 3, it is expectedly observed a general rise in power with increase in sample size in all the test statistics and in all the distributions considered except the standard multivariate normal. Under this standard multivariate normal distribution, it is expected that the power will be equal to the nominal α of 5%. This amounts to type I error rate since the null distribution of multivariate normality is true. The proposed $L^2_{n,d}$ test together with the $T_{n,d}$ and $G_{n,d}$ tests maintained this in all the sample sizes and variable dimensions considered showing that it has a very good control over type-1 error rate. Also, the proposed test failed to be generally more powerful than the rest of the competing tests especially at small sample sizes. However, at large sample sizes of more than 50, the power performances of the proposed test no doubt were generally at worst as good as any other competing technique in almost all the alternative distributions considered both at $d = 2$ and $d = 5$. Again, no much difference in the pattern of power behaviour of the proposed test was recorded at difference variable dimensions. The test therefore can be recommended as a very powerful technique for assessing multivariate normality of data sets especially at large sample sizes.

Table 2: Power comparison of tests for multivariate normality for various multivariate distributions at α = 5%, d = 2

n	**Distributions**	***HZ***	***Scl***	$T_{n,d}$	$G_{n,d}$	$L^2_{n,d}$
25	MVN	4.1	4.5	4.9	5.3	5.1
50	MVN	4.6	4.3	4.9	5.3	5.0
100	MVN	4.8	5.1	5.1	5.1	5.3
500	MVN	4.8	4.8	5.2	5.0	5.1
25	MVt	78.4	66.7	81.7	78.6	74.5
50	MVt	96.9	90.6	97.5	97.8	97.4
100	MVt	100.0	99.5	99.9	100.0	100.0
500	MVt	100.0	100.0	100.0	100.0	100.0
25	MVC	98.8	93.0	98.3	97.8	97.0
50	MVC	100.0	99.7	100.0	100.0	100.0
100	MVC	100.0	100.0	100.0	100.0	100.0
500	MVC	100.0	100.0	100.0	100.0	100.0
25	Arcsine^2	64.8	30.4	18.4	75.1	79.2
50	Arcsine^2	99.5	77.4	34.8	98.9	99.0
100	Arcsine^2	100.0	99.6	72.7	100.0	100.0
500	Arcsine^2	100.0	100.0	100.0	100.0	100.0
25	$t^2(2)$	75.6	62.5	79.4	74.5	69.9
50	$t^2(2)$	95.6	88.7	96.3	96.4	95.7
100	$t^2(2)$	99.9	99.1	99.9	100.0	99.9
500	$t^2(2)$	100.0	100.0	100.0	100.0	100.0
25	$\text{Beta}(1,5)^2$	72.6	29.8	62.7	24.9	20.7
50	$\text{Beta}(1,5)^2$	98.4	46.2	86.8	44.3	41.7
100	$\text{Beta}(1,5)^2$	100.0	66.8	97.8	67.9	66.5
500	$\text{Beta}(1,5)^2$	100.0	99.9	100.0	99.9	99.9
25	$\text{Exp}(1)^2$	93.6	57.4	86.6	56.9	51.6
50	$\text{Exp}(1)^2$	100.0	84.2	98.1	85.9	84.6
100	$\text{Exp}(1)^2$	100.0	97.8	100.0	98.6	98.5
500	$\text{Exp}(1)^2$	100.0	100.0	100.0	100.0	100.0
25	Mvgevd	38.5	28.3	40.9	26.7	23.4
50	Mvgevd	68.2	47.9	69.6	51.0	49.4
100	Mvgevd	93.4	70.8	94.2	76.6	75.9
500	Mvgevd	100.0	99.9	100.0	100.0	100.0
25	$\text{LN}(0,1)^2$	99.1	83.1	97.4	84.7	82.3
50	$\text{LN}(0,1)^2$	100.0	98.0	99.9	98.7	98.7
100	$\text{LN}(0,1)^2$	100.0	100.0	100.0	100.0	100.0
500	$\text{LN}(0,1)^2$	100.0	100.0	100.0	100.0	100.0
25	Beta(1,5) Exp(1)	85.5	45.9	94.8	41.3	36.4
50	Beta(1,5) Exp(1)	99.7	68.8	100.0	69.9	68.6
100	Beta(1,5) Exp(1)	100.0	90.4	100.0	93.1	92.2
500	Beta(1,5) Exp(1)	100.0	100.0	100.0	100.0	100.0
25	LN(0,1)Beta(1,5)	94.5	64.7	98.4	63.3	59.3
50	LN(0,1)Beta(1,5)	100.0	89.1	100.0	91.2	90.3
100	LN(0,1)Beta(1,5)	100.0	98.8	100.0	99.5	99.5
500	LN(0,1)Beta(1,5)	100.0	100.0	100.0	100.0	100.0

Table 3: Power comparison of tests for multivariate normality for various multivariate distributions at α = 5%, d = 5

n	Distributions	HZ	Scl	$T_{n,d}$	$G_{n,d}$	$L^2_{n,d}$
25	MVN	3.4	4.9	4.9	5.1	4.8
50	MVN	4.4	5.1	4.9	5.2	4.6
100	MVN	4.5	5.0	4.9	5.1	5.1
500	MVN	4.6	4.6	4.8	5.1	5.4
25	MVt	93.9	70.2	94.1	84.7	78.9
50	MVt	100.0	98.2	99.8	100.0	100.0
100	MVt	100.0	100.0	100.0	100.0	100.0
500	MVt	100.0	100.0	100.0	100.0	100.0
25	MVC	100.0	96.2	99.8	99.4	99.2
50	MVC	100.0	100.0	100.0	100.0	100.0
100	MVC	100.0	100.0	100.0	100.0	100.0
500	MVC	100.0	100.0	100.0	100.0	100.0
25	Arcsine[5]	27.9	10.3	3.2	67.3	69.7
50	Arcsine[5]	95.2	24.8	4.1	99.6	99.7
100	Arcsine[5]	100.0	60.0	15.5	100.0	100.0
500	Arcsine[5]	100.0	100.0	97.1	100.0	100.0
25	$t^5(2)$	83.1	60.9	90.4	70.8	62.6
50	$t^5(2)$	99.3	95.6	99.3	99.4	99.2
100	$t^5(2)$	100.0	100.0	100.0	100.0	100.0
500	$t^5(2)$	100.0	100.0	100.0	100.0	100.0
25	Beta(1,5)[5]	63.7	17.1	41.1	3.5	2.1
50	Beta(1,5)[5]	97.9	35.4	72.4	30.8	24.4
100	Beta(1,5)[5]	100.0	60.0	94.2	67.2	63.2
500	Beta(1,5)[5]	100.0	99.9	100.0	100.0	100.0
25	Exp(1)[5]	92.1	47.6	81.6	32.9	23.1
50	Exp(1)[5]	100.0	85.2	97.9	89.8	87.0
100	Exp(1)[5]	100.0	99.1	100.0	99.9	99.9
500	Exp(1)[5]	100.0	100.0	100.0	100.0	100.0
25	Mvgevd	28.6	18.3	49.1	6.7	4.1
50	Mvgevd	65.2	44.7	82.4	48.0	43.3
100	Mvgevd	95.5	75.1	98.4	86.2	85.2
500	Mvgevd	100.0	100.0	100.0	100.0	100.0
25	LN (0,1)[5]	99.8	84.1	98.4	83.1	76.2
50	LN (0,1)[5]	100.0	99.5	100.0	99.9	99.8
100	LN (0,1)[5]	100.0	100.0	100.0	100.0	100.0
500	LN (0,1)[5]	100.0	100.0	100.0	100.0	100.0
25	$Beta(1,5)^3$ $Exp(1)^2$	78.8	31.4	89.2	12.4	7.9
50	$Beta(1,5)^3$ $Exp(1)^2$	99.8	63.4	99.3	64.2	58.1
100	$Beta(1,5)^3$ $Exp(1)^2$	100.0	90.5	100.0	94.9	94.5
500	$Beta(1,5)^3$ $Exp(1)^2$	100.0	100.0	100.0	100.0	100.0
25	$LN(0,1)^3Beta(1,5)^2$	97.2	66.8	98.4	57.6	47.7
50	$LN(0,1)^3Beta(1,5)^2$	100.0	96.1	100.0	97.9	97.5
100	$LN(0,1)^3Beta(1,5)^2$	100.0	100.0	100.0	100.0	100.0
500	$LN(0,1)^3Beta(1,5)^2$	100.0	100.0	100.0	100.0	100.0

In order to make for a better comparison, the average power performances of these competing tests are obtained for each sample size and number of variables involved. Often times in comparisons of this nature, consideration has always been given to the nature of the alternative distribution, in terms of symmetry, see Thulin [13] for instance. However in practice, the nature of the distribution whose multinormality is in question is usually not known. As a result, the symmetry of the distributions is ignored and the average powers of the competing tests on all the 11 distributions considered are obtained as presented in Table 4. The power plots in terms of the average powers are also presented in Figures1 and 2 respectively for the number of variables 2 and 5.

From Table 4 and Figures 1 and 2, the newly proposed test gives a competitive power performance. Its power performance appears to be inferior to other tests at small sample size of 25 except for the S_{cl} test. However, it maintained a progressive power gain along the direction of increasing sample size that at n = 500, its powers came top in averagely both variable dimensions.

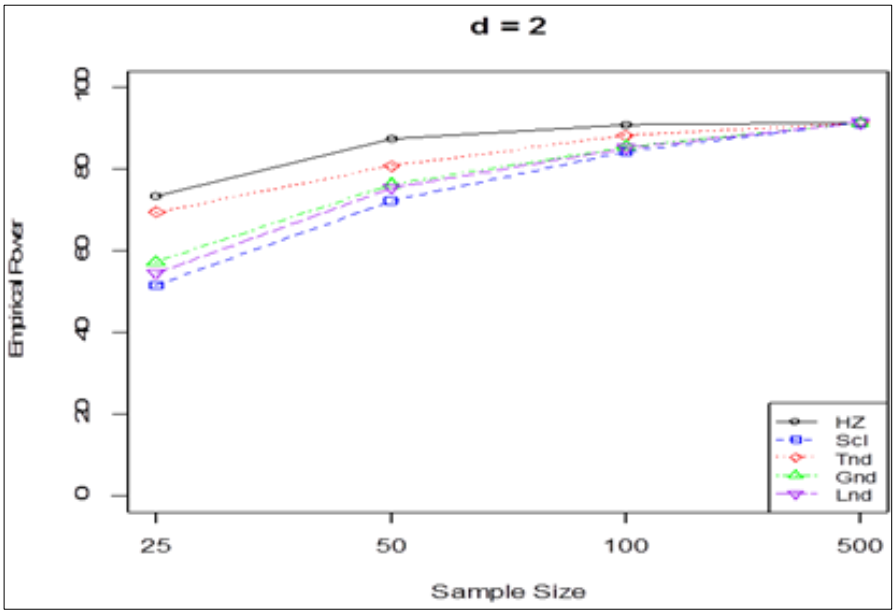

Fig 1: Average power performances of five tests for multivariate normality at sample sizes n = 25, 50, 100 and 500, d = 2 and α = 0.05.

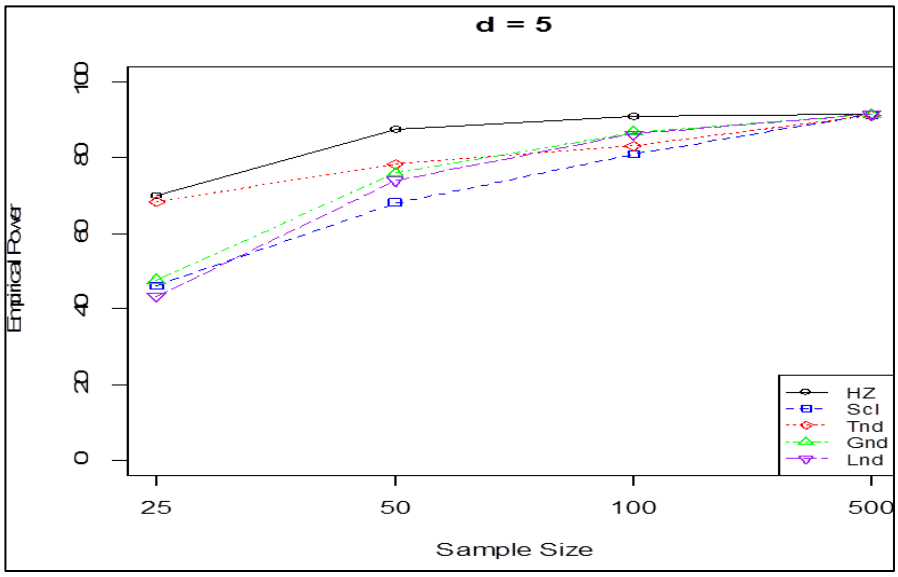

Fig 2: Average power performances of five tests for multivariate normality at sample sizes n = 25, 50, 100 and 500, d = 5 and α = 0.05

Table 4: Average power performances of tests for multivariate normality for various multivariate distributions at α = 5%, d = 2, 5

d	n	HZ	S_{cl}	$T_{n,d}$	$G_{n,d}$	$L^2_{n,d}$
1	25	73.23	51.48	69.41	57.19	54.49
	50	87.54	72.26	80.72	76.31	75.49
	100	90.75	84.35	88.15	85.53	85.25
	500	91.35	91.33	91.38	91.35	91.36
2	25	69.86	46.16	68.20	47.59	43.30
	50	87.44	68.00	78.19	75.89	73.96
	100	90.91	80.88	83.00	86.66	86.17
	500	91.33	91.32	91.08	91.37	91.40

5. Application of the $L^2_{n,d}$ test to the breast cancer data

In order to show the applicability of the proposed test to real-life data sets, a 500 observation vectors with 5-component data set was extracted from a 32-component breast cancer diagnostic data set, retrieved from www.world/datasets/multivariate [14]. Each variable in the data set represents a characteristic of the cell nuclei present in images of a fine needle aspirate (*FNA*) of a breast mass. In order to test for multivariate normality of the extracted data set, an R statistical code for the L_2 statistic gave the value of the statistic as 15835.78 against the critical value of the statistics at α = 0.05 with n = 500 and d = 5 which is 72.8074. This shows clearly that the data set is not multivariate normal at α = 0.05. The result of the test is supported with the quantile-quantile plot of the chi-squared transformed form of the data, which is expected to be linear if the data set is actually multivariate normal, see Figure 3 below.

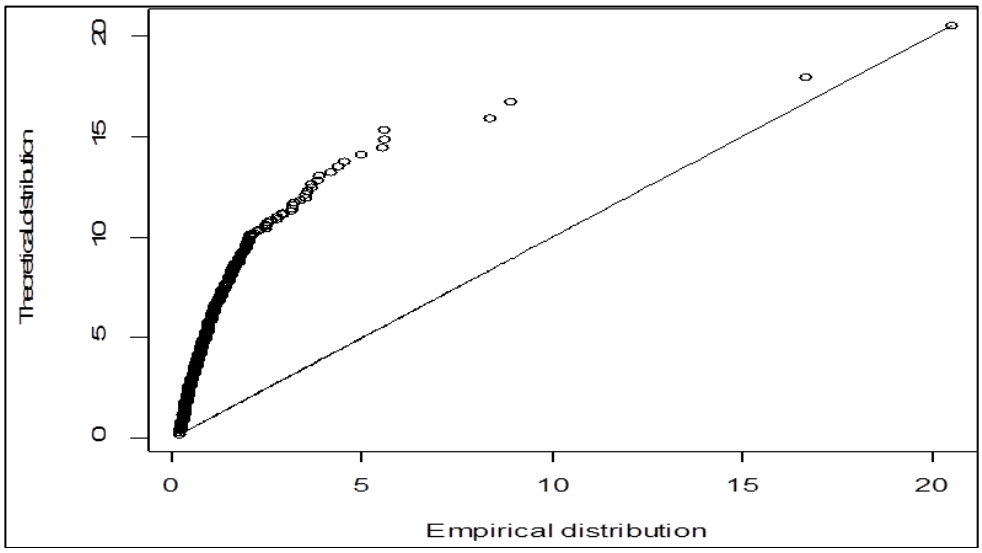

Fig 3: Quantile-quantile plot of the breast cancer diagnostic data

6. Conclusion

In this paper, it has been shown that an equivalent chi – square transformation of the multivariate normal data sets can be obtained by the principle of Euclidean distance instead of the traditional Mahalanobis distance. As a result, the $L^2_{n,d}$ statistic has been proposed for testing the assumption of *MVN*. The proposed statistic is an alternative to the $G_{n,d}$ statistic according to Madukaife and Okafor [11]. From the power comparisons carried out in this work, the proposed $L^2_{n,d}$ test, no doubt, can be recommended for use as a good tool for testing multinormality, having a very strong control over type-I error as well as highly competitive power.

7. References

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