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Ramesh Kumar Joshi
Department of Statistics, Trichandra Multiple Campus, Kathmandu, Nepal

Vijay Kumar
Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Corresponding Author:
Vijay Kumar
Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Lindley Gompertz distribution with properties and applications

Ramesh Kumar Joshi and Vijay Kumar

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Abstract
In this article, we have presented a new three-parameter probability distribution called the Lindley Gompertz distribution. The proposed model can exhibit increasing, increasing-decreasing, j-shaped, and bathtub, and a broad variety of monotone failure rates. Some properties of this model are studied and calculated. Three extensively used estimation methods are used to estimate the model parameters namely maximum likelihood estimators (MLE), least-square (LSE), and Cramer-Von-Mises (CVM) methods. By using the maximum likelihood estimate we have constructed the asymptotic confidence interval for the model parameters. All the computations are performed in R software. The potentiality of the proposed distribution is revealed by using some graphical methods and statistical tests taking a real dataset, where the proposed distribution provided a better fit and more flexible in comparison with some other lifetime distributions.

Keywords: Lindley distribution, gompertz distribution, LSE, CVE

1. Introduction

Most of the continuous probability distributions have been introduced in recent years but the real data sets related to life sciences, engineering, finance, climatology, medicine, geology, biology, hydrology, ecology, reliability, life testing, and risk analysis do not provide a better fit to these distributions. So, the creation of new modified distributions seems to be necessary to address the problems in these fields. The generalized, extended, and modified distributions are created by adding one or more parameters or performing some transformations to the baseline distribution. Therefore, the new proposed distributions provide the best fit and more flexible compared to the competing models. Researchers in the last few decades have developed various extensions and a modified form of the Lindley distribution which was developed by (Lindley, 1958) [19] in the context of Bayesian statistics, as a counterexample to fiducial statistics. An extensive study on the Lindley distribution was done by (Ghitany *et al*., 2008) [11], Mazucheli & Achcar (2011) [21] have been used the Lindley distribution to competing risks lifetime data.

A random variable X follows Lindley distribution with parameter θ and its probability density function (PDF) is given by

$$f_L(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};\ x>0,\theta>0 \tag{1}$$

And its cumulative density function (CDF) is

$$F_L(x) = 1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x};\ x>0,\theta>0 \tag{2}$$

In this article, we put forward the new Lindley Gompertz (LG) distribution to enhance the capability of the Lindley distribution using the Lindley-G family by inserting only one extra parameter. It is a member of the Lindley-G family introduced by Ristic and Balakrishnan (2012) [27].

Here, we have taken the Lindley distribution as a generator and the Gompertz as a baseline distribution. Towards the modified theoretical distribution, Ghitany *et al*. (2011) [12] introduced weighted Lindley distribution having two parameters and has shown that it is appropriate in modeling survival data for a mortality study. Nadarajah *et al*. (2011) [24] have introduced generalized Lindley, extended Lindley by (Bakouch *et al*., 2012) [4], Ashour and Eltehiwy (2015) [3] for exponentiated power Lindley, Bhati *et al*. (2015) [5] Lindley–Exponential distribution. Ieren *et al*. (2018) [15] have introduced modeling lifetime data with Weibull-Lindley distribution. Ibrahim *et al*. (2019) [14] has introduced a new extension of Lindley distribution. Alizadeh *et al*. (2020) [2] have introduced the odd log-logistic Lindley-G family of distributions and illustrated the Bayesian and non-Bayesian estimation of the model parameter.

Cakmakyapan and Ozel (2016) [7] have presented a new class of distributions to generate new distribution based on the Lindley generator (Lindley-G) having additional shape parameter θ. The CDF and PDF of Lindley-G are respectively,

$$F(\varphi;\theta,\lambda) = 1-\left[1-G(\varphi;\lambda)\right]^{\theta}\left[1-\frac{\theta}{\theta+1}\ln\left\{1-G(\varphi;\lambda)\right\}\right];\ \varphi>0,\theta>0 \tag{3}$$

And

$$f(\varphi;\theta,\lambda) = \frac{\theta^{2}}{\theta+1}g(\varphi;\lambda)\left[\overline{G}(\varphi;\lambda)\right]^{\theta-1}\left[1-\ln\overline{G}(\varphi;\lambda)\right];\ \varphi>0,\theta>0 \tag{4}$$

Where

$$g(\varphi;\lambda) = \frac{dG(\varphi;\lambda)}{d\varphi}$$, and $$\overline{G}(\varphi;\lambda) = 1-G(\varphi;\lambda)$$

The Gompertz distribution is one of the classical probability distribution that represents survival function based on laws of mortality. This distribution performs a significant role in modeling human mortality and analyzing actuarial tables. The Gompertz distribution was first introduced by (Gompertz, 1824) [13]. It has been used as a growth model and also used to fit the tumor growth. The Gompertz function reduced a significant collection of data in life tables into a single function. It is based on the assumption that the mortality rate decreases exponentially as a person ages. The resulting Gompertz function is for the number of individuals living at a given age as a function of age. Applications and an extensive survey of the Gompertz distribution can be found in (Ahuja & Nash, 1967) [1]. Cooray and Ananda (2010) [9] have introduced the Gompertz-sinh family and it was used to analyze the survival data with highly negatively skewed distribution. El-Gohary *et al*. (2013) [10] have presented a flexible called the generalized Gompertz distribution it has increasing or constant or decreasing or bathtub curve failure rate depending upon the shape parameter. Ieren *et al*. (2019) [16] have introduced a three-parameter power Gompertz distribution using a power transformation approach.

The motivation of this study is to obtain a more flexible model by adding just one extra parameter to the Gompertz distribution to achieve a better fit to the real data. We study the properties of the LG distribution and explore its applicability.

The contents of this article are organized as follows. The new Gompertz distribution is introduced and various distributional properties are discussed in Section 2. Three commonly used estimation methods are employed to estimate the model parameters namely maximum likelihood estimators (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods, further, the maximum likelihood estimators are used to construct the asymptotic confidence intervals using the observed information matrix is discussed in Section 3. In Section 4 a real data sets have been taken to investigate the applications and suitability of the proposed distribution. In this section, we present the ML estimators of the parameters and approximate confidence intervals also AIC, BIC, AICC, HQIC are calculated to assess the validity of the LG model. Finally, Section 5 ends up with some general concluding remarks.

2. The Lindley Gompertz distribution

The Gompertz distribution was first introduced by (Gompertz, 1824) [13]. Let X be a random variable follows the Gompertz distribution with parameters α and λ if its cumulative distribution function can be written as,

$$G(x) = 1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp(\alpha x)\right)\right\}\ ;\ \alpha>0,\lambda>0,\ x>0 \tag{5}$$

and its corresponding probability density function can be expressed as,

$$g(x) = \lambda\exp\left\{\alpha x+\frac{\lambda}{\alpha}\left(1-\exp(\alpha x)\right)\right\};\ \alpha>0,\lambda>0,\ \theta>0,\ x>0 \tag{6}$$

Using (5) and (6) as CDF and PDF of baseline distribution to (3) and (4) we get the CDF and PDF of Lindley Gompertz distribution with parameters $(\alpha,\ \lambda,\ \theta)>0$ respectively as

$$F(x) = \left(1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp(\alpha x)\right)\right\}\right)^{\theta}\left[1-\left(\frac{\theta}{1+\theta}\right)\log\left(1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp(\alpha x)\right)\right\}\right)\right];\ \ x>0 \tag{7}$$

and

$$f(x) = \lambda \left(\frac{\theta^2}{1+\theta}\right) e^{\alpha x+\delta} \left(1-e^{\delta}\right)^{\theta-1} \left\{1-\left(\frac{\theta}{1+\theta}\right)\log\left(1-e^{\delta}\right)\right\} \tag{8}$$

Where

$$\delta = \frac{\lambda}{\alpha}\left(1-e^{\alpha x}\right)$$

2.1 Reliability/survival function

The reliability function of Lindley Gompertz (LG) distribution is

$$R(x) = 1 - F(x)$$

$$\mathrm{R(x)} = \left[1-\left(\frac{\theta}{1+\theta}\right)\left(1-e^{(\lambda x)^{\alpha}}\right)\right]\exp\left[\theta\left(1-e^{(\lambda x)^{\alpha}}\right)\right];\ \alpha,\lambda,\theta>0,\ x>0 \tag{9}$$

2.2. Hazard function

The failure rate function of LG distribution can be defined as,

$$\mathrm{h(x)} = \frac{f(x)}{R(x)} = \frac{\alpha\lambda^{\alpha}\left(\frac{\theta^2}{1+\theta}\right)x^{\alpha-1}e^{(\lambda x)^{\alpha}}\left[1-\left(1-e^{(\lambda x)^{\alpha}}\right)\right]}{\left[1-\left(\frac{\theta}{1+\theta}\right)\left(1-e^{(\lambda x)^{\alpha}}\right)\right]};\ \alpha,\lambda,\theta>0,\ x>0 \tag{10}$$

2.3 Quantile function

The quantile function of LG (α, λ, θ) can be expressed as,

$$\mathrm{p\text{-}1} + \left[1-\left(\frac{\theta}{1+\theta}\right)\left(1-e^{(\lambda x)^{\alpha}}\right)\right]\exp\left[\theta\left(1-e^{(\lambda x)^{\alpha}}\right)\right] = 0,\ 0<\mathrm{p}<1 \tag{11}$$

2.4 The Random Deviate Generation:

The random deviate can be generated from LG (α, λ, θ) by

$$\mathrm{v\text{-}1} + \left[1-\left(\frac{\theta}{1+\theta}\right)\left(1-e^{(\lambda x)^{\alpha}}\right)\right]\exp\left[\theta\left(1-e^{(\lambda x)^{\alpha}}\right)\right] = 0,\ 0<\mathrm{v}<1 \tag{12}$$

Solving (12) for x we get the expression for the random deviate generation; here *v* has the *U* (0, 1) distribution.
Plots of probability density function and hazard rate function of LG (α, λ, θ) with different values of parameters are shown in Figure 1.

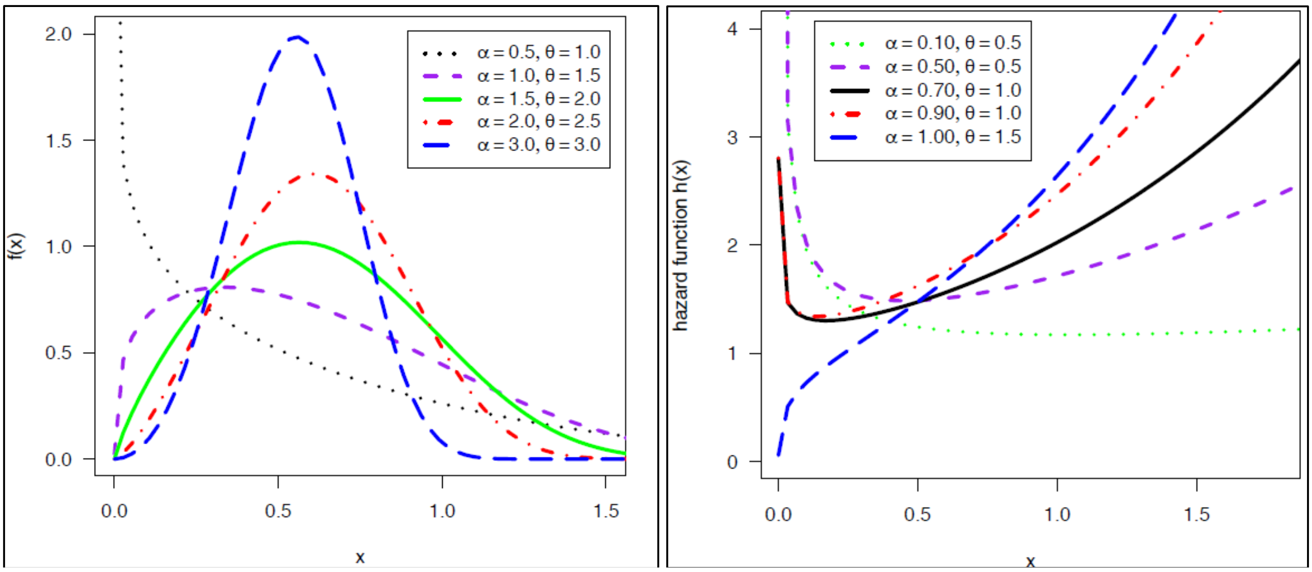

Fig 1: Plots of density function (left panel) and hazard function (right panel) for different values of α, λ and θ.

2.5 Skewness and Kurtosis of LG distribution:
In descriptive statistics, the measures of skewness and kurtosis play a significant role in data analysis. The coefficient of Bowley's skewness measure based on quartiles is given by

$$Skewness(B) = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}, \tag{13}$$

and the coefficient of Moor's kurtosis measures based on octiles Moors (1988) is given by

$$K(moors) = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}, \tag{14}$$

3. Methods of Estimation
The objective of estimation is to approximate the value of a model parameter based on sample information. The estimation theory deals with the basic problem of inferring some relevant features of a random experiment based on the observation of the experiment outcomes.
There are so many methods for estimating unknown parameters of the model. We have considered three types of estimation methods such as the maximum likelihood (MLE), ordinary least squares (LSE), and the Cramer-von Mises (CVM) method.

3.1 Maximum Likelihood Estimation (MLE)
We have illustrated the maximum likelihood estimators (MLE's) of the LG (α, λ, θ) distribution. Let $\underset{\sim}{x} = (x_1, \ldots, x_n)$ be the observed values of size 'n' from LG (α, λ, θ) then the likelihood function for the parameter vector $\Delta = (\alpha, \lambda, \theta)^T$ can be written as,

$$L(\Delta) = \lambda \left(\frac{\theta^2}{1+\theta}\right) \prod_{i=1}^{n} e^{\alpha x_i + \delta_i} \left(1 - e^{\delta_i}\right)^{\theta - 1} \left\{1 - \left(\frac{\theta}{1+\theta}\right) \log\left(1 - e^{\delta_i}\right)\right\}$$ taking logarithms on both sides we get

$$\ln L(\Delta) = n \ln \alpha + \alpha n \ln \alpha + 2n \ln \theta - n \ln(1+\theta) + (\alpha - 1)\sum_{i=1}^{n} \ln x_i + \\ (\lambda x_i)^{\alpha} + \theta \sum_{i=1}^{n} \left(1 - e^{(\lambda x_i)^{\alpha}}\right) + \sum_{i=1}^{n} \ln\left\{1 - \left(1 - e^{(\lambda x_i)^{\alpha}}\right)\right\} \tag{15}$$

The elements of the score function $B(\Delta) = (B_\alpha, B_\lambda, B_\theta)$ are obtained as

$$B_\alpha = \frac{n}{\alpha} + n \ln \lambda + \sum_{i=1}^{n} \ln x_i + 2\sum_{i=1}^{n} (\lambda x_i)^{\alpha} \ln(\lambda x_i) - \theta \sum_{i=1}^{n} (\lambda x_i)^{\alpha} e^{(\lambda x_i)^{\alpha}} \ln(\lambda x_i)$$

$$B_\lambda = \frac{n\alpha}{\lambda} + \frac{2\alpha}{\lambda}\sum_{i=1}^{n} (\lambda x_i)^{\alpha} - \frac{\alpha\theta}{\lambda}\sum_{i=1}^{n} (\lambda x_i)^{\alpha} e^{(\lambda x_i)^{\alpha}} \tag{16}$$

$$B_\theta = n + \frac{n(2+\theta)}{\theta(\theta+1)} - \sum_{i=1}^{n} e^{(\lambda x_i)^{\alpha}}$$

Equating B_α, B_λ and B_θ to zero and solving these non-linear equations simultaneously gives the MLE $\hat{\Delta} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$ of $\Delta = (\alpha, \lambda, \theta)^T$. Manually we cannot solve these equations so by using the computer software R, Mathematica, Matlab, or any other programs and Newton-Raphson's iteration method, one can solve these equations.

Let us denote the parameter vector by $\Delta = (\alpha, \lambda, \theta)^T$ and the corresponding MLE of Δ as $\hat{\Delta} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$, then the asymptotic normality results in, $(\hat{\Delta} - \Delta) \rightarrow N_3\left[0, (I(\Delta))^{-1}\right]$ where $I(\Delta)$ is the Fisher's information matrix given by,

$$I(\Delta) = -\begin{pmatrix} E(B_{\alpha\alpha}) & E(B_{\alpha\lambda}) & E(B_{\alpha\theta}) \\ E(B_{\alpha\lambda}) & E(B_{\lambda\lambda}) & E(B_{\lambda\theta}) \\ E(B_{\alpha\theta}) & E(B_{\lambda\theta}) & E(B_{\theta\theta}) \end{pmatrix}$$

Further differentiating (16) we get,

$$B_{\alpha\alpha} = -\frac{n}{\alpha^2} - \theta\sum_{i=1}^{n}(\lambda x_i)^{2\alpha} e^{(\lambda x_i)^\alpha}\left[\ln(\lambda x_i)\right]^2 - \theta\sum_{i=1}^{n}(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha}\left[\ln(\lambda x_i)\right]^2$$

$$B_{\lambda\lambda} = \frac{\alpha}{\lambda^2}\sum_{i=1}^{n}\left[\theta(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha} - 2(\lambda x_i)^{\alpha} - 1\right] - \frac{\alpha^2}{\lambda^2}\sum_{i=1}^{n}\left[\theta(\lambda x_i)^{2\alpha} e^{(\lambda x_i)^\alpha} + \theta(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha} - 2(\lambda x_i)^{\alpha}\right]$$

$$B_{\theta\theta} = \frac{n}{(\theta+1)^2} - \frac{2n}{\theta^2}$$

$$B_{\alpha\lambda} = \frac{n}{\lambda} + \frac{2}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{\alpha} - \frac{\theta}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha} - \frac{\alpha\theta}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{2\alpha} e^{(\lambda x_i)^\alpha}\ln(\lambda x_i)$$
$$- \frac{\theta}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha}\ln(\lambda x_i) + \frac{2}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{\alpha}\ln(\lambda x_i)$$

$$B_{\alpha\theta} = -\sum_{i=1}^{n}(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha}\ln(\lambda x_i)$$

$$B_{\lambda\theta} = -\frac{\alpha}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^{\alpha} e^{(\lambda x_i)^\alpha}$$

The observed fisher information matrix $A(\hat{\Delta})$ as an estimate of the information matrix $I(\Delta)$ given by

$$A(\hat{\Delta}) = -\begin{pmatrix} B_{\alpha\alpha} & B_{\alpha\lambda} & B_{\alpha\theta} \\ B_{\alpha\lambda} & B_{\lambda\lambda} & B_{\lambda\theta} \\ B_{\alpha\theta} & B_{\lambda\theta} & B_{\theta\theta} \end{pmatrix}\Bigg|_{(\hat{\alpha},\hat{\lambda},\hat{\theta})} = -H(\Delta)\big|_{(\Delta=\hat{\Delta})}$$

here H is the Hessian matrix.

The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H(\Delta)\big|_{(\Delta=\hat{\Delta})}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\alpha},\hat{\theta}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) & \mathrm{cov}(\hat{\theta},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\theta}) & \mathrm{cov}(\hat{\theta},\hat{\lambda}) & \mathrm{var}(\hat{\theta}) \end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, λ, and θ can be constructed as, $\hat{\alpha} \pm Z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\alpha})}$, $\hat{\lambda} \pm Z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\lambda})}$ and $\hat{\theta} \pm Z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\theta})}$, where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

3.2 Method of Least-Square Estimation (LSE)

The ordinary least square estimators and weighted least square estimators are introduced by (Swain *et al*., 1988) [26] to estimate the parameters of Beta distributions. The least-square estimators of the unknown parameters for α, λ, and θ of LG distribution can be obtained by minimizing

$$D(X;\alpha,\lambda,\theta)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2 \tag{17}$$

with respect to unknown parameters α, λ, and θ.

Let $F(X_{(i)})$ denotes the cumulative distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \ldots\ldots..< X_{(n)}$, where $\{X_1, X_2, \ldots\ldots.., X_n\}$ is a random sample of size n from a (2.3). Therefore, the least square estimators of α, λ, and θ say $\hat{\alpha}, \hat{\lambda}$ and $\hat{\theta}$ respectively, can be obtained by minimizing

$$D(X;\alpha,\lambda,\theta)=\sum_{i=1}^{n}\left[\left(1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp\left(\alpha x_i\right)\right)\right\}\right)^{\theta}\left[1-\left(\frac{\theta}{1+\theta}\right)\log\left(1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp\left(\alpha x_i\right)\right)\right\}\right)\right]-\frac{i}{n+1}\right]^2 \tag{18}$$

with respect to α, λ, and θ.

To obtain the least square estimates of α, λ, and θ, we have to solve the following two nonlinear equations simultaneously by equating to zero,

3.3 Method of Cramer-Von-Mises (CVM)

We are interested in Cramér-von-Mises type minimum distance estimators, (Macdonald 1971) [20] because it provides empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of α, λ, and θ are obtained by minimizing the function

$$H_{CVM}(\alpha,\beta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\mid\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^2 \tag{19}$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left(1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp\left(\alpha x_i\right)\right)\right\}\right)^{\theta}\left[1-\left(\frac{\theta}{1+\theta}\right)\log\left(1-\exp\left\{\frac{\lambda}{\alpha}\left(1-\exp\left(\alpha x_i\right)\right)\right\}\right)\right]-\frac{2i-1}{2n}\right]^2;\ x>0$$

To obtain the CVM estimators of α, λ, and θ, we have to solve the following two nonlinear equations simultaneously by equating to zero.

4. Application with a real dataset

In this section, we illustrate the applicability of LIW distribution using a real dataset used by earlier researchers. Lawless (2003) [18] failures can occur in microcircuits because of the movement of atoms in the conductors in the circuit; this is referred to as electromigration. The data below are from an accelerated life test of 59 conductors. Failure times are in hours and there are no censored observations.

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923

The maximum likelihood estimates are calculated directly by using optim() function in R software (R Core Team, 2020) [27] and (Ming Hui, 2019) [22] by maximizing the likelihood function (3.1). We have obtained $\hat{\alpha}$ = 0.1765, $\hat{\beta}$ = 0.2051, $\hat{\theta}$ = 11.4574 and corresponding Log-Likelihood value is -111.8046. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, λ, and θ.

Table 1: MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	0.1765	0.0466	(0.0853, 0.2678)
lambda	0.2051	0.0594	(0.0887, 0.3215)
theta	11.4574	3.1947	(5.1959, 17.7189)

The plots of profile log-likelihood function (Kumar & Ligges, 2011) [17] for the parameters α, λ and θ have been displayed in Figure 2 and noticed that the ML estimates can be uniquely determined.

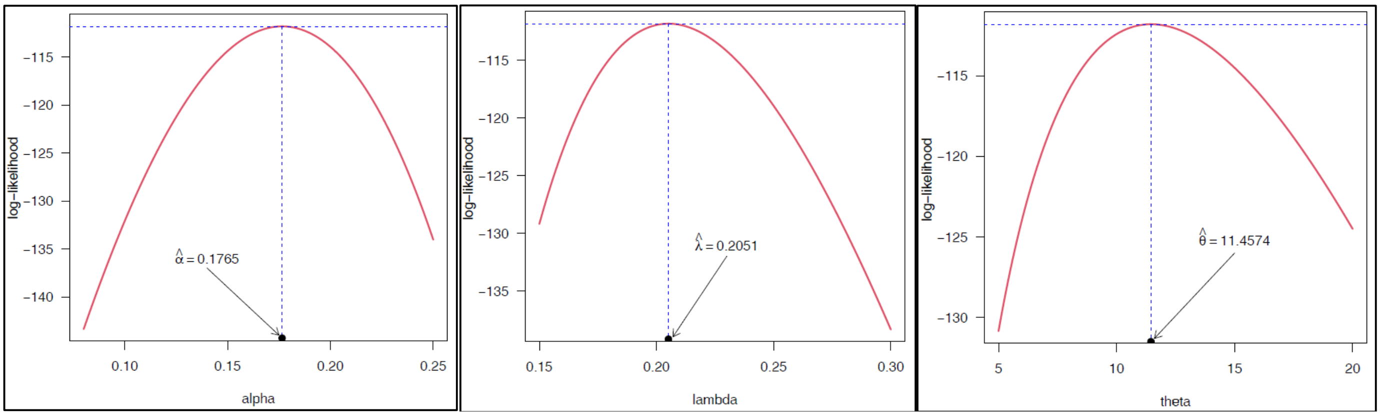

Fig 2: Graph of Profile log-likelihood function for the parameters α, λ and θ.

In Table 2 we have displayed the estimated value of the parameters of Lindley Gompertz distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood, AIC, BIC, AICC and HQIC criterion. In Table 3 we have presented The KS, W and A^2 statistics with their corresponding p-value of MLE, LSE and CVE method.

Table 2: Estimated parameters, log-likelihood, AIC, BIC, AICC and HQIC

Method of Estimation	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	-LL	AIC	BIC	AICC	HQIC
MLE	0.17654	0.20507	11.4574	111.8046	229.6092	235.8418	230.0455	232.0421
LSE	0.20892	0.17088	10.37019	112.0091	230.0182	236.2508	230.4545	232.4511
CVE	0.15321	0.26034	17.3537	112.3254	230.6508	236.8834	231.0872	233.0838

Table 3: The KS, W and A^2 statistics with a p-value

Method of Estimation	*KS(p-value)*	*W(p-value)*	*A^2M(p-value)*
MLE	0.0620(0.9665)	0.0331(0.9661)	0.1988(0.9908)
LSE	0.0553(0.9893)	0.0273(0.9848)	0.2125(0.9866)
CVE	0.0516(0.9953)	0.0256(0.9890)	0.2457(0.9726)

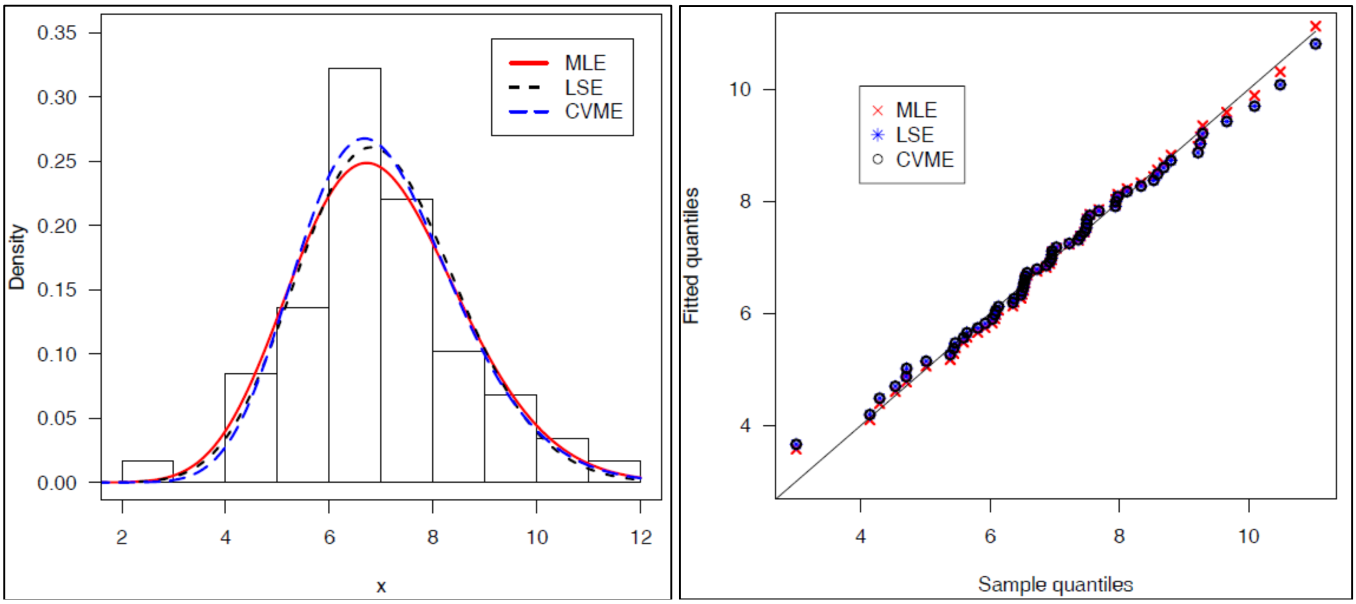

Fig 3: The Histogram and the density function of fitted distributions of estimation methods MLE, LSE and CVM (left panel) and fitted quantiles (right panel).

To illustrate the goodness of fit of the Lindley Gompertz distribution, we have taken some well-known distribution for comparison purpose which are listed blew,

Flexible Weibull Extension (FWE) distribution

The density of Flexible Weibull (FW) distribution (Bebbington, 2007) [5] with parameters α and β is

$$f_{FW}(x) = \left(\alpha + \frac{\beta}{x^2}\right) \exp\left(\alpha x - \frac{\beta}{x}\right) \exp\left\{-\exp\left(\alpha x - \frac{\beta}{x}\right)\right\} \quad ; x \geq 0, \alpha > 0, \beta > 0.$$

Lindley-Exponential (LE) distribution
The probability density function of LE (Bhati, 2015) [6] can be expressed as

$$f_{LE}(x) = \lambda\left(\frac{\theta^2}{1+\theta}\right)e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}\left\{1-\ln\left(1-e^{-\lambda x}\right)\right\};\ \lambda,\theta>0,\ x>0$$

Exponential power (EP) distribution
The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) [25] is

$$f_{EP}(x) = \alpha\,\lambda^{\alpha}\,x^{\alpha-1}\,e^{(\lambda x)^{\alpha}}\exp\left\{1-e^{(\lambda x)^{\alpha}}\right\}\ ;(\alpha,\lambda)>0,\quad x\ge 0 .$$

where α and λ are the shape and scale parameters, respectively.

Gompertz distribution (GZ)
The probability density function of Gompertz distribution (Murthy *et al*., 2003) [23] with parameters α and θ is

$$f_{GZ}(x) = \theta\ e^{\alpha x}\ exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha x}\right)\right\}\quad ;x\ge 0,\theta>0,-\infty<\alpha<\infty .$$

In Figure 4 we have plotted the Q-Q plot and CDF plot and it is seen that the proposed distribution fits the data very well.

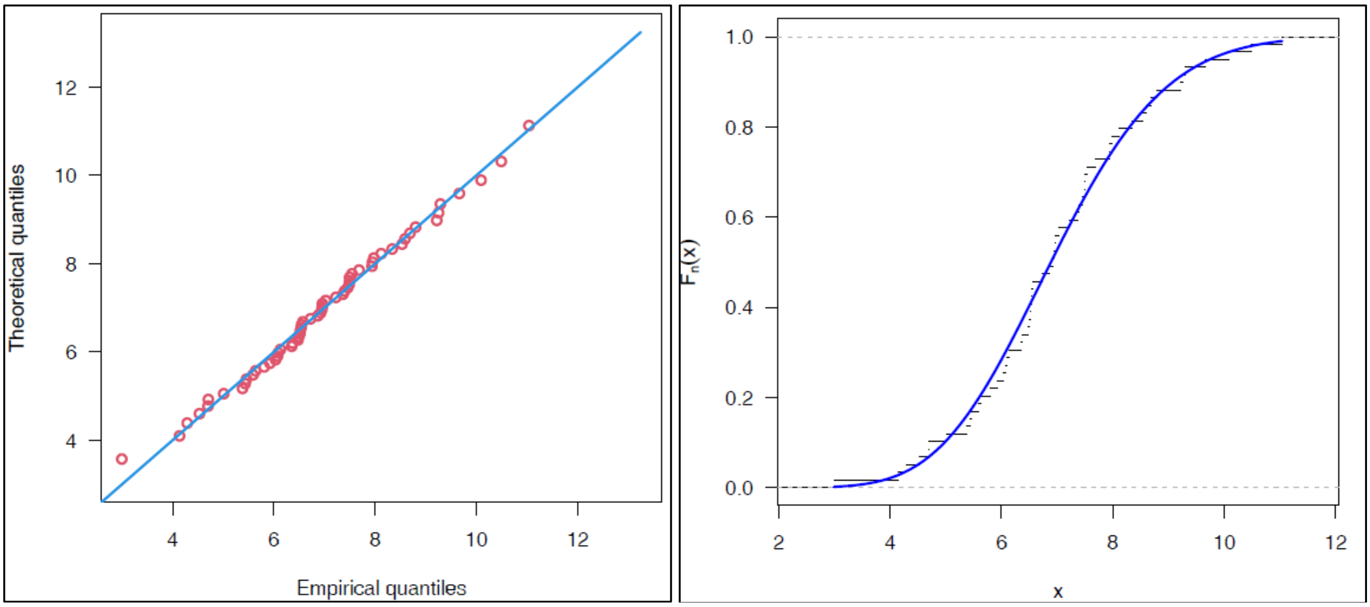

Fig 4: The P-P plot (left panel) and CDF plot (right panel) of the LG distribution.

For the assessment of potentiality of the proposed model we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
LG	111.8046	229.6092	235.8418	230.0455	232.0421
FW	112.2883	228.5766	232.7317	228.7909	230.1986
LE	114.9528	233.9055	238.0606	234.1198	235.5275
EP	116.5015	237.0029	241.1580	237.2098	238.6249
GZ	117.1740	238.3480	242.5031	238.5623	239.9700

The Histogram and the density function of fitted distributions and Empirical distribution function with the estimated distribution function of LG and some selected distributions are presented in Figure 5.

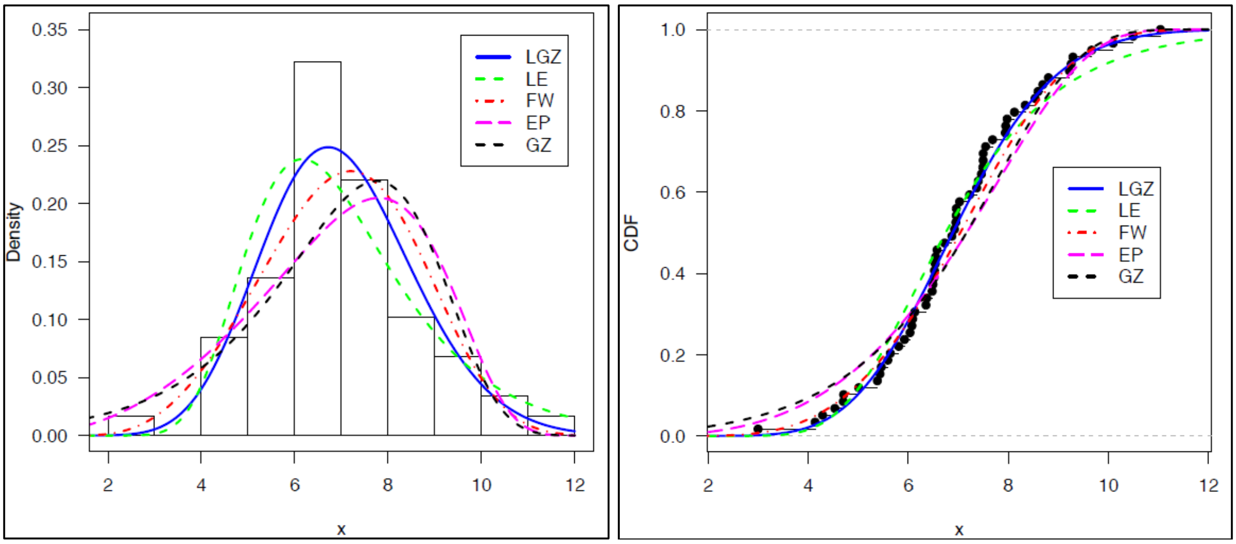

Fig 5: The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the LG distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistics in Table 5. It is observed that the LG distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the LG distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*W(p-value)*	*A^2M(p-value)*
LG	0.0620 (0.9665)	0.0331 (0.9661)	0.1988 (0.9908)
FW	0.0971 (0.5991)	0.0858 (0.6606)	0.4686 (0.7783)
LE	0.1042 (0.5099)	0.1173 (0.5077)	0.7373 (0.5279)
EP	0.1365 (0.2021)	0.2398 (0.2021)	1.3735 (0.2098)
GZ	0.1306 (0.2464)	0.216 (0.2387)	1.3143 (0.2277)

5. Conclusion

In this article, we have developed the three-parameter continuous Lindley Gompertz (LG) distribution. For our study, we have provided the statistical and mathematical properties of the proposed model. The shape of the PDF of the LG model is uni-modal and positively skewed and the hazard function can exhibit increasing, increasing-decreasing, j-shaped and bathtub, and a broad variety of monotone failure rates. The P-P and KS plots showed that the purposed distribution is quite better for fitting the real dataset. Finally, using a real data set we have explored some well-known estimation methods namely maximum likelihood estimation (MLE), least-square (LSE), and Cramer-Von-Mises (CVM) methods. Further, we also construct the asymptotic confidence interval for MLEs. We conclude that MLE is the best estimation method as compared to the LSE and CVM methods. The application illustrates that the proposed model provides a consistently better fit and more flexible than other underling models. We expect that this model will contribute to the field of survival analysis.

6. References

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