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The logistic-Rayleigh distribution with properties and applications

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Abstract
In this article, we have presented a two-parameter univariate continuous distribution called Logistic-Rayleigh distribution. We have discussed some mathematical and statistical properties of the distribution, such as the cumulative distribution function, probability density function and hazard rate function, survival function, quantile function, the skewness, and kurtosis measures. The model parameters of the proposed distribution are estimated using three well-known estimation methods, namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods. The goodness of fit of the purposed distribution is also evaluated by fitting it in comparison with some other existing distributions using a real data set.

Keywords: Cramer-von-mises estimation, least-square estimation, logistic distribution, rayleigh distribution, survival function

1. Introduction

In most of the literature of probability distributions and applied statistics, it is observed that the study of reliability and survival analysis in various fields of applied statistics and life sciences, the probability distributions are often used. In modelling survival data, existing models do not always reveal a better fit. Hence most of the researchers are interested in generalizing standard distributions and investigating their flexibility and applicability. Usually, these new compounded models produce an improved fit as compared to usual classical survival models. They are obtained by bringing one or more additional shape parameter(s) to the parent distribution. The exponential distribution plays a vital role in analyses of life testing data in statistics and probability theory. It is the probability distribution of the time between events in a Poisson point process, i.e., a process in which events occur independently and continuously at a constant average rate. It is a specific case of the gamma distribution. It is the continuous analogue of the geometric distribution, and it has the key property of being memory less. In addition to being used for the analysis of Poisson point processes, it is found in various other contexts. A compounded survival model that includes different shapes like increasing, decreasing, bathtub-shaped, and inverted Bathtub-Shaped failure rate in a single model would be constructive in survival analysis. Such a model would provide considerable flexibility and goodness of fit for fitting a broad variety of lifetime data sets. Such a survival model might also be taken to determine the distribution class from which the data is selected by constructing a confidence interval over its parameters. The proposed distribution introduced here satisfies these criteria. The Rayleigh was obtained from the amplitude of sound resulting from many important sources by (Rayleigh, 1880) [18]. It is a continuous probability distribution with a wide range of applications such as in reliability analysis, life testing experiments, clinical studies and applied statistics. This distribution is a particular case of the two-parameter Weibull distribution with the shape parameter equal to 2. Important features, including its origin, can be found in the work of (Siddiqui, 1962) [19], as well as Howlader and Hossian (1995). A non-negative random variable X follows the Rayleigh distribution with scale parameter θ if its probability density function (PDF) is given by

$$f(x;\theta) = \theta x e^{-\frac{\theta}{2}x^2}; \theta > 0 \tag{1.1}$$

And its corresponding cumulative distribution function (CDF) is given as

$$F(x;\theta)=1-e^{-\frac{\theta}{2}x^2};\quad \theta>0 \tag{1.2}$$

Continuation to the broad variety of applications associated with the Rayleigh distribution, many authors have created different modifications of the distribution which have led to some flexible and good distributions such as the generalized Rayleigh distribution by (Kundu & Raqab, 2005) [5], Bivariate generalized Rayleigh distribution by (Abdel-Hady, 2013) [1], Transmuted Rayleigh distribution by (Merovci, 2013) [12], generalized Weibull-Rayleigh distribution by (Yahaya & Alaku, 2018) [23], New Lindley-Rayleigh distribution by (Joshi & Kumar, 2020) [7], Weibull-Rayleigh distribution by (Merovci & Elbatal, 2015) [11], transmuted Weibull-Rayleigh distribution by (Yahaya & Ieren, 2017) [24] as well as the Transmuted Inverse Rayleigh distribution studied by (Ahmad *et al*., 2015) [2].

The logistic distribution is a univariate continuous distribution, and both its PDF and CDF functions have been used in many different areas such as logistic regression, logit models and neural networks. It has been used in the physical sciences, demography, sports modelling, and recently in finance as well as insurance. The logistic distribution has wider tails than a normal distribution, so it is more consistent with the underlying data and provides better insight into the likelihood of extreme events.

Let X be a non-negative random variable which follows the logistic distribution with shape parameter $\theta > 0$, and then its cumulative distribution function is given by

$$G(x;\theta)=\frac{1}{1+e^{-\theta x}};\quad \theta>0, x\in\Re \tag{1.3}$$

and its corresponding PDF is

$$g(x;\theta)=\frac{\theta e^{-\theta x}}{\left(1+e^{-\theta x}\right)^2};\quad \theta>0, x\in\Re \tag{1.4}$$

Tahir *et al*. (2016) [22] has defined the *logistic-X family* as a new generating family of continuous distributions produced from a logistic random variable whose density function can be defined as being right-skewed, left-skewed, symmetrical and reversed-J shaped, and can have decreasing, increasing, bathtub and upside-down bathtub hazard rates shaped. Lan and Leemis (2008) [10] have presented an approach to define the logistic compounded model and introduced the logistic–exponential survival distribution. This has several useful probabilistic properties for lifetime modelling. Unlike most distributions in the bathtub and upside-down bathtub classes, the logistic–exponential distribution exhibit closed-form density, hazard, cumulative hazard, and survival functions. The survival function of logistic–exponential distribution is

$$S(x;\lambda)=\frac{1}{1+\left(e^{\lambda x}-1\right)^{\alpha}};\quad \alpha>0,\lambda>0, x\geq 0 \tag{1.5}$$

Using the same approach used by (Lan & Leemis, 2008) [10] we have defined the new distribution called Logistic-Rayleigh (L-R) distribution. The main objective of this article is to introduce a more flexible distribution by inserting just one extra parameter to the Rayleigh distribution to attain a better fit for the lifetime data sets. We have discussed some distributional properties and their applicability. The different sections of the proposed study are arranged as follows. In Section 2, we present the Logistic- Rayleigh (L-R) distribution and its various mathematical and statistical properties. We have made use of three well-known estimation methods to estimate the model parameters, namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. In Section 4, a real data set has been analyzed to explore the applications and capability of the proposed distribution. In this section, we present the estimated value of the parameters and log-likelihood, AIC, BIC and AICC criterion for ML, LSE, and CVME also the goodness of fit of the proposed distribution is also evaluated by fitting it in comparison with some other existing distributions using a real data set. Finally, in Section 5, some concluding remarks are presented.

2. The logistic-rayleigh (l-r) distribution

Using the same approach used by (Lan & Leemis, 2008) [10] we have defined the new distribution called Logistic-Rayleigh (L-R) distribution. Let X be a non-negative random variable with a positive shape parameter α and a positive scale parameter λ then CDF of logistic-Rayleigh distribution can be defined as

$$F(x)=1-\frac{1}{1+\left(e^{(\lambda x^2/2)}-1\right)^{\alpha}}\quad ;x\geq 0,\alpha>0,\lambda>0. \tag{2.1}$$

The density of logistic-Rayleigh (LR) distribution with shape parameter α and scale parameter λ is

$$f(x)=\frac{\lambda\,\alpha\,x\,e^{(\lambda x^2/2)}\left(e^{(\lambda x^2/2)}-1\right)^{\alpha-1}}{\left\{1+\left(e^{(\lambda x^2/2)}-1\right)^{\alpha}\right\}^2}\quad ;x\geq 0,\alpha>0,\lambda>0. \tag{2.2}$$

This CDF function be similar to the log-logistic CDF function with the second term of the denominator being changed in its base to a Rayleigh function; hence we named it Logistic-Rayleigh distribution.

A. Reliability function

The reliability function of Logistic- Rayleigh distribution is

$$R(x)=1-F(x)$$

$$=\frac{1}{1+\left(e^{(\lambda x^2/2)}-1\right)^{\alpha}}\quad ;x\geq 0,\alpha>0,\lambda>0. \tag{2.3}$$

B. Hazard function

The failure rate function of L-R distribution can be defined as,

$$h(x)=\frac{f(x)}{R(x)}=\frac{\lambda\,\alpha\,x\,e^{(\lambda x^2/2)}\left(e^{(\lambda x^2/2)}-1\right)^{\alpha-1}}{1+\left(e^{(\lambda x^2/2)}-1\right)^{\alpha}};x\geq 0,\alpha>0,\lambda>0. \quad (2.4)$$

In Figure 1, we have displayed the plots of the PDF and hazard rate function of L-R distribution for different values of α and λ.

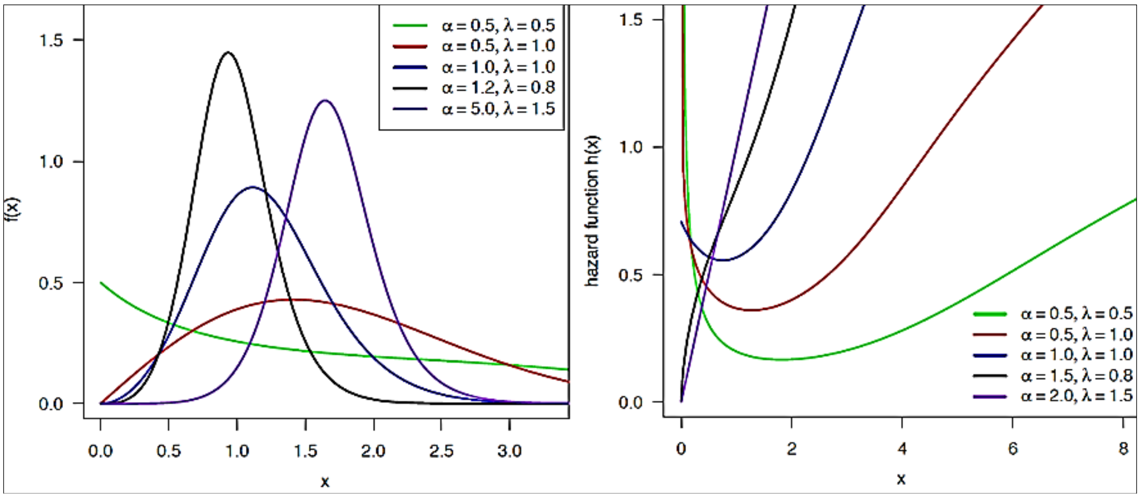

Fig 1: Plots of PDF (upper panel) and hazard function (lower panel) for different values of α and λ.

C. Quantile function

Quantile function of Logistic - Rayleigh distribution can be expressed as

$$Q(u)=\left[\frac{2}{\lambda}\log\left\{1+\left(\frac{u}{1-u}\right)^{1/\alpha}\right\}\right]^{1/2} \quad ;0<u<1. \quad (2.5)$$

D. Median

The median of Logistic-Rayleigh distribution can be expressed as

$$Median=\left[\frac{2}{\lambda}\log 2\right]^{1/2}$$

Here we notice that the median depends upon scale parameter λ only.

E. Skewness and Kurtosis

The measures Skewness and Kurtosis based on quantiles are, Bowley's coefficient of skewness is

$$\kappa(B)=\frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(0.75)-Q(0.25)}, \quad (2.6)$$

And coefficient of kurtosis based on octiles which were defined by (Moors, 1988) [14] is

$$K_u(M)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}, \quad (2.7)$$

3. Methods of estimation

In this section, the parameters of the proposed distribution are estimated by applying some well-known estimation methods which are as follows

3.1. Maximum Likelihood Estimation Method

For the estimation of the parameter, the maximum likelihood method is the most commonly used method (Casella & Berger, 1990). Let, $x_1,x_2,...,x_n$ is a random sample from Logistic-Rayleigh (α, λ) and the likelihood function, $L(\alpha,\lambda)$ is given by,

$$L(\tau;x_1,x_2,...,x_n)=f(x_1,x_2,...,x_n/\tau)=\prod_{i=1}^{n}f(x_i/\tau)$$

$$L(\alpha,\lambda)=\alpha\lambda\prod_{i=1}^{n}\frac{x_i\,e^{(\lambda x_i^2/2)}\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha-1}}{\left\{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}\right\}^2};x\geq 0,\alpha>0,\lambda>0.$$

Now log-likelihood density is

$$\ell=n\log\lambda+n\log\alpha+\sum_{i=1}^{n}\log x_i+\frac{\lambda}{2}\sum_{i=1}^{n}x_i^2+(\alpha-1)\sum_{i=1}^{n}\log\left(e^{(\lambda x_i^2/2)}-1\right)-2\sum_{i=1}^{n}\log\left\{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}\right\} \quad (3.1.1)$$

Differentiating (3.1.1) with respect to α and λ we get,

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(e^{(\lambda x_i^2/2)}-1\right)\left(1-\left\{e^{(\lambda x_i^2/2)}-1\right\}^{\alpha}\right)\left(1+\left\{e^{(\lambda x_i^2/2)}-1\right\}^{\alpha}\right)^{-1} \quad (3.1.2)$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}+\frac{1}{2}\sum_{i=1}^{n}x_i^2+\frac{(\alpha-1)}{2}\sum_{i=1}^{n}\frac{x_i^2\,e^{(\lambda x_i^2/2)}}{\left(e^{(\lambda x_i^2/2)}-1\right)}-\alpha\sum_{i=1}^{n}\frac{x_i^2\,e^{(\lambda x_i^2/2)}\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha-1}}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}} \quad (3.1.3)$$

Equating (3.1.2) and (3.1.3) to zero and solving simultaneously for α and λ, we get the maximum likelihood estimate $\hat{\alpha}$ *and* $\hat{\lambda}$ of the parameters α and λ. By using computer software like R, Matlab, Mathematica etc. for maximization of (3.1.1) we can obtain the estimated value of α and λ. For the confidence interval estimation of α and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α and λ can be obtained as,

$$D=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

Where

$$D_{11}=\frac{\partial^2 l}{\partial\alpha^2},\quad D_{22}=\frac{\partial^2 l}{\partial\lambda^2} \quad \text{and} \quad D_{12}=D_{21}=\frac{\partial^2 l}{\partial\alpha\partial\lambda}$$

Let $\Xi=(\alpha,\lambda)$ denote the parameter space and the corresponding MLE of Ξ as $\hat{\Xi}=(\hat{\alpha},\hat{\lambda})$, then $\left(\hat{\Xi}-\Xi\right)\to N_2\left[0,\left(D(\Xi)\right)^{-1}\right]$ where $D(\Xi)$ is the Fisher's information matrix. Using the Newton-Raphson algorithm to maximize the likelihood creates the observed information matrix, and hence the variance-covariance matrix is obtained as,

$$\left[D(\Xi)\right]^{-1}=\begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} \tag{3.1.4}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α and λ can be constructed as,

$$\hat{\alpha}\pm z_{\alpha/2}SE(\hat{\alpha}) \quad \text{and} \quad \hat{\lambda}\pm z_{\alpha/2}SE\left(\hat{\lambda}\right) \tag{3.1.5}$$

where $z_{\alpha/2}$ is the upper percentile of standard normal variate.

3.2. Method of Least-Square Estimation (LSE)

Swain *et al.* (1988) [21] proposed the weighted least square estimators and ordinary least square estimators to estimate the parameters of Beta distributions. Here we have applied the same procedure for the L-R distribution. The least-square estimators of the unknown parameters α and λ of L-R distribution can be obtained by minimizing

$$D(X;\alpha,\lambda)=\sum_{i=1}^{n}\left[G(X_i)-\frac{i}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters α and λ.

Consider $G(X_i)$ denotes the distribution function of the ordered random variables $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ where $\{X_1,X_2,\ldots,X_n\}$ is a random sample of size n from a distribution function G(.). The least-square estimators of α and λ say $\hat{\alpha}$ *and* $\hat{\lambda}$ respectively, can be obtained by minimizing

$$D(X;\alpha,\lambda)=\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}}-\frac{i}{n+1}\right]^2;x\geq 0,\alpha>0,\lambda>0. \tag{3.2.2}$$

with respect to α and λ.

Differentiating (3.2.2) with respect to α and λ we get,

$$\frac{\partial D}{\partial\alpha}=-2\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha}\ln\left[e^{(\lambda x_i^2/2)}-1\right]}{\left\{1+\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial D}{\partial\lambda}=-2\alpha\lambda\sum_{i=1}^{n}x_i\left[1-\frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha-1}e^{(\lambda x_i^2/2)}}{\left\{1+\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha}\right\}^2}$$

Similarly, the weighted least square estimators can be obtained by minimizing

$$D(X;\alpha,\lambda)=\sum_{i=1}^{n}w_i\left[G(X_{(i)})-\frac{i}{n+1}\right]^2$$

with respect to α and λ. The weights w_i are

$$w_i=\frac{1}{Var(X_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$$

Hence, the weighted least square estimators of α and λ respectively can be obtained by minimizing,

$$D(X;\alpha,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}}-\frac{i}{n+1}\right]^2 \tag{3.2.3}$$

With respect to α and λ.

3.3. Method of Cramer-Von-Mises estimation (CVME)

The CVME estimators of α and λ are obtained by minimizing the function

$$W(X;\alpha,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[G\left(x_{i:n}\,|\,\alpha,\lambda\right)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]^2 \tag{3.3.1}$$

Differentiating (3.3.1) with respect to α and λ we get,

$$\frac{\partial W}{\partial\alpha}=-2\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]\frac{\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha}\ln\left[e^{(\lambda x_i^2/2)}-1\right]}{\left\{1+\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial W}{\partial \lambda} = -2\alpha\lambda \sum_{i=1}^{n} x_i \left[1 - \frac{1}{1+\left(e^{(\lambda x_i^2/2)}-1\right)^{\alpha}} - \frac{2i-1}{2n} \right] \left[\frac{\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha-1} e^{(\lambda x_i^2/2)}}{\left\{1+\left[e^{(\lambda x_i^2/2)}-1\right]^{\alpha}\right\}^2} \right]$$

Solving $\frac{\partial W}{\partial \alpha} = 0 \text{ and } \frac{\partial W}{\partial \lambda} = 0$ simultaneously we will get the CVM estimators.

Illustration with a real dataset

The data presented below are from an accelerated life test of 59 conductors used by (Nelson & Doganaksoy, 1995) [16]. The failures can occur in microcircuits because of the movement of atoms in the conductors in the circuit; this is referred to as electro-migration. The failure times are in hours, and there are no censored observations.

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

The contour plot and fitted CDF with empirical distribution function (EDF) are presented in Figure 2, Kumar & Ligges (2011) [8].

The MLEs are calculated directly by using optim() function in R software (R Core Team, 2020) [17] and (Ming, 2019) [13] by maximizing the likelihood function (3.1.1). We have obtained $\hat{\alpha}$ = 2.6967 and $\hat{\lambda}$ = 0.0291 and the corresponding Log-Likelihood value is -111.2045. In Table 1, we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α and λ.

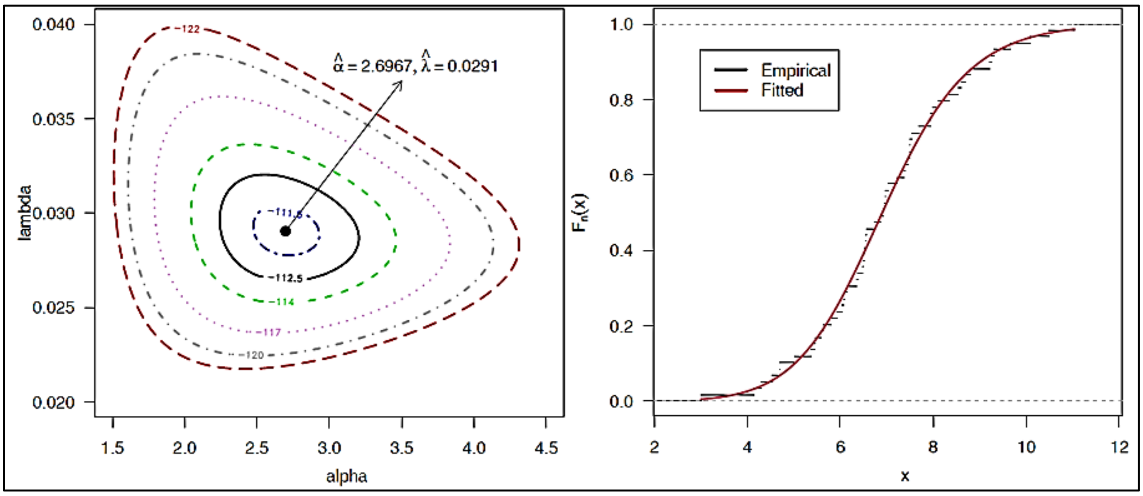

Fig 2: Contour plot (upper panel) and the fitted CDF with empirical distribution function (lower panel) of L-R distribution

Table 1: MLEs, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	2.6967	0.2983	(2.1121, 3.2814)
lambda	0.0291	0.0018	(0.0257, 0.0324)

We have displayed the graph of the profile log-likelihood function of α and λ in Figure 3 and observed that the MLEs are unique. By using MLE, LSE and CVME methods we estimate the parameter of L-R distribution.

For the goodness of fit purpose, we useAkaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information Criterion (AICC) and Hannan-Quinn information criterion (HQIC), negative log-likelihood (-LL), a statistic to select the best model among selected models. The expressions to calculate AIC, BIC, AICC and HQIC are listed below:

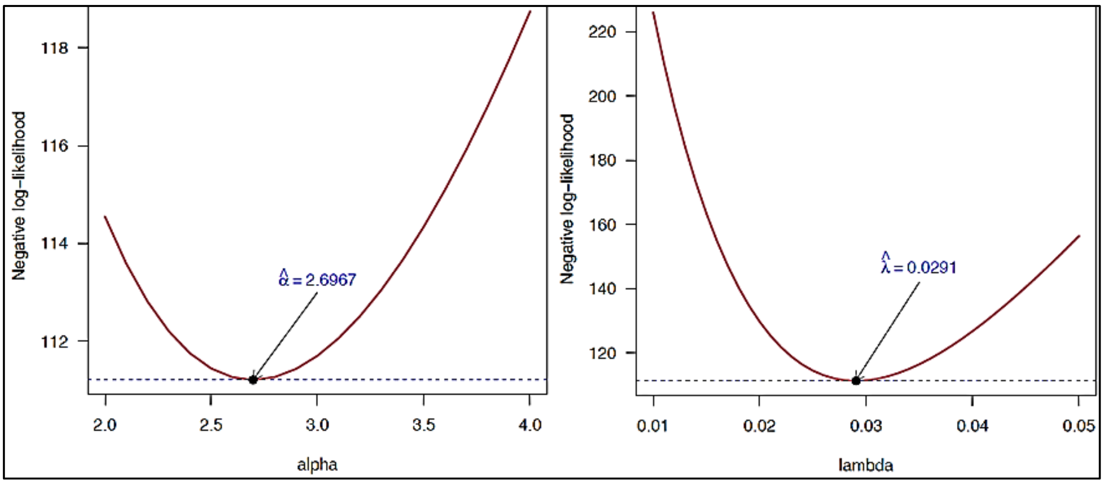

Fig 3: Graph of profile log-likelihood function of α and λ.

a) $AIC = -2l(\hat{\theta}) + 2k$

b) $BIC = -2l(\hat{\theta}) + k\log(n)$

c) $AICC = AIC + \frac{2k(k+1)}{n-k-1}$

d) $HQIC = -2l(\hat{\theta}) + 2k\log[\log(n)]$

where the number of parameters is denoted by k and n denotes sample size in the model under consideration. Further, in order to evaluate the fits of the LHC distribution with some selected distributions, we have taken the Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistic. These statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. These statistics are calculated as

$$KS = \max_{1\le i\le n}\left(d_i - \frac{i-1}{n}, \frac{i}{n} - d_i\right)$$

$$W = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i + \ln(1-d_{n+1-i})\right]$$

$$A^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - d_i\right]^2$$

Where

$d_i = CDF(x_i)$; the x_i's being the ordered observations.

In Table 2 we have displayed the estimated value of the parameters of Logistic inverse exponential distribution using MLE, LSE and CVME method and their corresponding negative log-likelihood, AIC, BIC, AICC and HQIC information criteria.

Table 2: Estimated parameters, log-likelihood, AIC, BIC, AICC and HQIC

Method of Estimation	$\hat{\alpha}$	$\hat{\lambda}$	-LL	AIC	BIC	AICC	HQIC
ML	2.6967	0.0291	111.2049	2 26.4098	230.5649	226.6241	228.0318
LSE	2.6617	0.0291	111.2115	2 26.423	230.578	226.6372	228.0449
CVME	2.7387	0.0291	111.2156	26.4313	230.5864	226.6456	228.0532

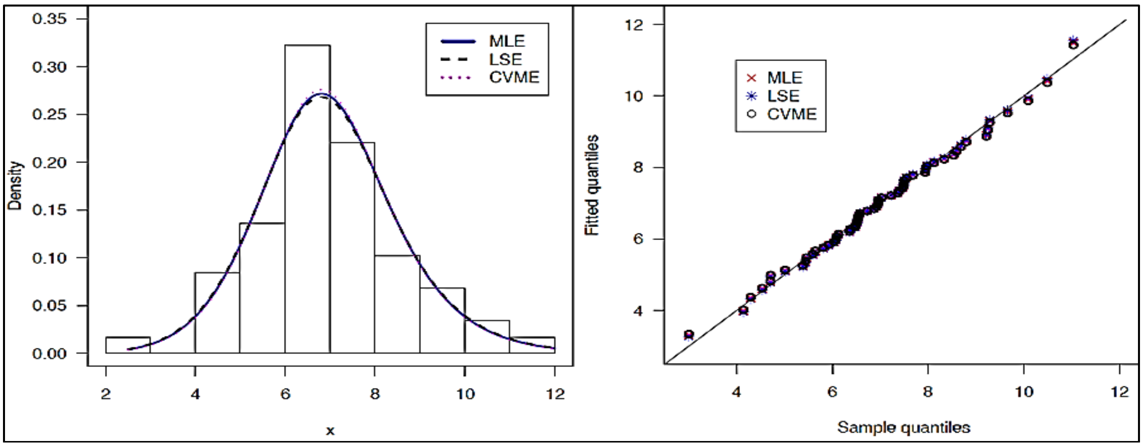

Fig 4: The Histogram and the density function of fitted distributions (upper panel) and Q-Q plot of estimation methods MLE, LSE and CVME (lower panel)

Table 3: The KS, AD and CVM statistic with p-value

Method of Estimation	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
MLE	0.0479(0.9983)	0.0219(0.9953)	0.1370(0.9994)
LSE	0.0497(0.9972)	0.0227(0.9942)	0.1370(0.9994)
CVME	0.0480(0.9982)	0.0215(0.9957)	0.1425(0.9991)

In order to illustrate the goodness of fit of the Lindley inverse exponential distribution, we have taken some well-known distribution for comparison purpose which are listed blew,

A. Logistic-Exponential (LE) distribution

The density of logistic-exponential (LE) distribution given by (Lan & Leemis, 2008) [10] with shape parameter α and scale parameter λ is

$$f_{LE}(x) = \frac{\lambda\,\alpha\, e^{\lambda x}\left(e^{\lambda x}-1\right)^{\alpha-1}}{\left\{1+\left(e^{\lambda x}-1\right)^{\alpha}\right\}^2} \quad ; x\ge 0,\ \alpha>0,\ \lambda>0.$$

B. Flexible Weibull Extension (FW) distribution

The density of Flexible Weibull (FW) distribution (Bebbington *et al*., 2007) with parameters α and β is

$$f_{FW}(x) = \left(\alpha + \frac{\beta}{x^2}\right)\exp\left(\alpha x - \frac{\beta}{x}\right)\exp\left\{-\exp\left(\alpha x - \frac{\beta}{x}\right)\right\} \quad ; x\ge 0, \alpha>0, \beta>0.$$

C. Generalized Exponential (GE) distribution

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999) [9]

$$f_{GE}(x;\alpha,\lambda) = \alpha\,\lambda\, e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1} ; (\alpha,\lambda)>0, x>0$$

D. Exponential power (EP) distribution

The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) is

$$f_{EP}(x) = \alpha\,\lambda^{\alpha}\, x^{\alpha-1}\, e^{(\lambda x)^{\alpha}} \exp\left\{1 - e^{(\lambda x)^{\alpha}}\right\} \quad ; (\alpha, \lambda) > 0, \quad x \ge 0$$

.

Where
α and λ are the shape and scale parameters, respectively.

E. Gompertz distribution(GZ)

The probability density function of Gompertz distribution (Murthy *et al*., 2003) with parameters α and θ is

$$f_{GZ}(x) = \theta\, e^{\alpha x} \exp\left\{\frac{\theta}{\alpha}\left(1 - e^{\alpha x}\right)\right\} \quad ; x \ge 0, \theta > 0, -\infty < \alpha < \infty.$$

In Figure 5,

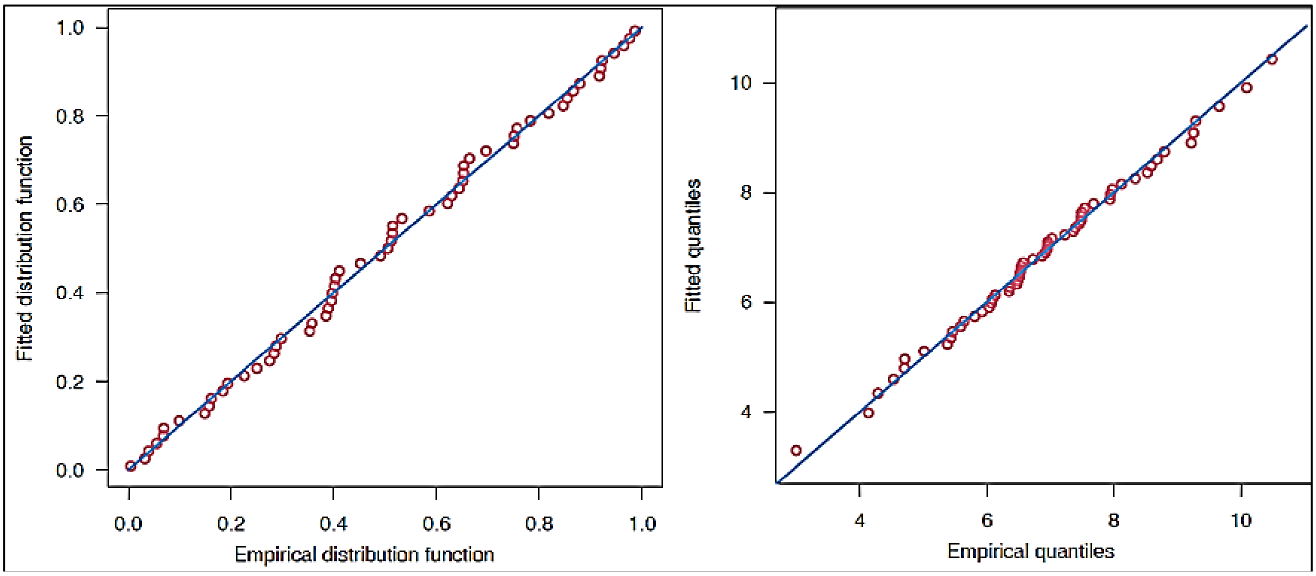

Fig 5: The P-P plot (upper panel) and Q-Q plot (lower panel) of L-R distribution

we have presented the P-P plot (empirical distribution function against theoretical distribution function) and Q-Q plot (empirical quantile against theoretical quantile).
For the judgment of potentiality of the proposed model, we have presented the value of Bayesian information criterion (BIC), Akaike information criterion (AIC), Corrected Akaike information criterion (AICC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Log-likelihood (LL), AIC, BIC, AICC and HQIC

Model	-LL	AIC	BIC	AICC	HQIC
L-R	111.2049	226.4098	230.5649	226.6241	228.0318
LogisExp	111.5138	227.0276	231.1827	227.2419	228.6496
FW	112.2883	228.5766	232.7317	228.7909	230.1986
GE	114.9473	233.8946	238.0497	234.1089	235.5166
EP	116.5015	237.0029	241.1580	237.2098	238.6249
GZ	117.1740	238.3480	242.5031	238.5623	239.9700

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of L-R and some selected distributions are presented in Figure 6.

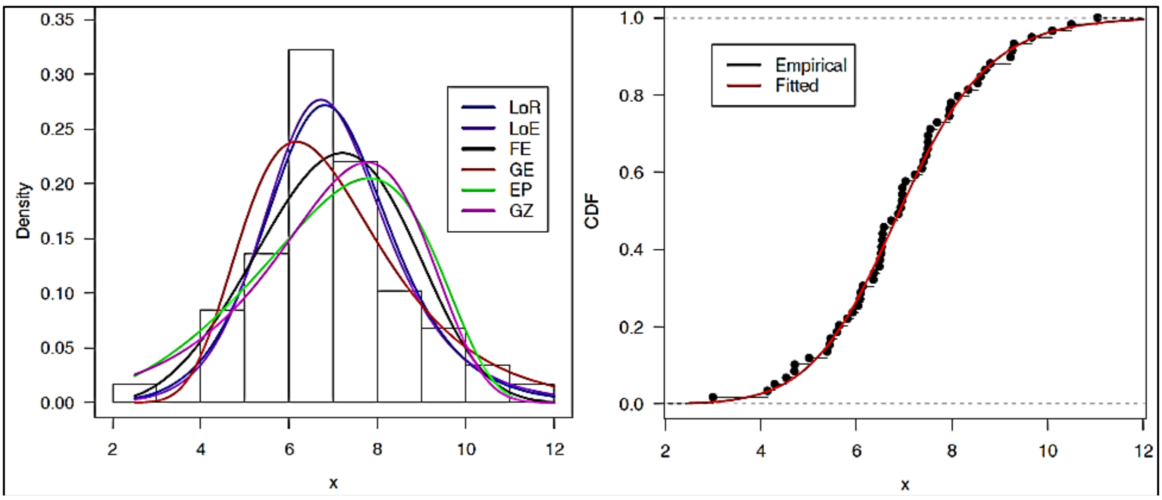

Fig 6: The Histogram and the density function of fitted distributions (upper panel) and Empirical distribution function with estimated distribution function (lower panel).

To compare the goodness-of-fit of the L-R distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the L-R distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the L-R distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
L-R	0.0479(0.9983)	0.0219(0.9953)	0.1370(0.9994)
LogisExp	0.0492(0.9975)	0.0203(0.9969)	0.0770(0.9998)
FW	0.0971(0.5991)	0.0858(0.6606)	0.4686(0.7783)
GE	0.1042(0.5103)	0.1173(0.5079)	0.7368(0.5282)
EP	0.1365(0.2021)	0.2398(0.2021)	1.3735(0.2098)
GZ	0.1304(0.2464)	0.2160(0.2387)	1.3143(0.2277)

Conclusion

In this study, we have introduced a two-parameter univariate continuous distribution named Logistic-Rayleigh (L-R) distribution. Some mathematical and statistical properties of the L-R distribution are discussed such as the shapes of the cumulative distribution function, probability density function and hazard rate function, survival function, hazard function quantile function, the skewness, and kurtosis measures are derived and established and found that the proposed model is flexible and inverted bathtub shaped hazard function. The model parameters are estimated by using three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods, and we concluded that the MLEs are quite better than LSE, and CVM. A real data set is considered to explore the applicability and suitability of the proposed distribution and found that the proposed model is quite better than other lifetime model taken into consideration. We hope this model may be an alternative in the field of survival analysis, probability theory and applied statistics.

References

1. Abdel-Hady DH. Bivariate generalized Rayleigh distribution. J of Appl. Sci. Res 2013;9(9):5403-5411.
2. Ahmad A, Ahmad SP, Ahmed A. Transmuted Inverse Rayleigh distribution: a generalization of the Inverse Rayleigh distribution. Math. Theo. & Mod 2015;4(7):90-98.
3. Bebbington M, Lai CD, Zitikis R. A flexible Weibull extension. Reliability Engineering & System Safety 2007;92(6):719-726.
4. Casella G, Berger RL. Statistical Inference. Brooks/ Cole Publishing Company, California 1990.
5. Gupta RD, Kundu D. Generalized exponential distributions, Australian and New Zealand Journal of Statistics 1999;41(2):173-188.
6. Howlader HA, Hossain A. On Bayesian estimation and prediction from Rayleigh based on type II censored data. Communications in Statistics-Theory and Methods 1995;24(9):2251-2259.
7. Joshi RK, Kumar V. New Lindley-Rayleigh Distribution with Statistical properties and Applications. International Journal of Mathematics Trends and Technology (IJMTT) 2020;66(9):197-208.
8. Kumar V, Ligges U, Relia R. A package for some probability distributions 2011. http://cran.r-project.org/web/packages/reliaR/index.html
9. Kundu D, Raqab MZ. Generalized Rayleigh distribution: different methods of estimations. Comp. Stat. & Data Anal 2005;49:187-200.
10. Lan Y, Leemis LM. The logistic–exponential survival distribution. Naval Research Logistics (NRL) 2008;55(3):252-264.
11. Merovci F, Elbatal I. Weibull Rayleigh Distribution: Theory and Applications. Appl. Math. & Inf. Sci 2015;9(5):1-11.
12. Merovci F. The transmuted Rayleigh distribution. Australian J of Stat 2013;22(1):21-30.
13. Ming Hui EG. Learn R for appL-Rd statistics. Springer, New York 2019.
14. Moors JJA. A quantile alternative for kurtosis. Journal of the Royal Statistical Society: Series D (The Statistician) 1988;37(1):25-32.
15. Murthy DNP, Xie M, Jiang R. Weibull Models, Wiley, New York 2003.
16. Nelson W, Doganaksoy N. Statistical analysis of life or strength data from specimens of various sizes using the power-(log) normal model. Recent Advances in Life-Testing and Reliability 1995,377-408.
17. R Core Team R. A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria 2020. URL https://www.R-project.org/.
18. Rayleigh L. Xii. on the resultant of a large number of vibrations of the same pitch and of arbitrary phase. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 1880;10(60):73-78.
19. Siddiqui MM. Some problems connected with Rayleigh distributions. Journal of Research of the National Bureau of Standards D 1962;66:167-174.
20. Smith RM, Bain LJ. An exponential power life-testing distribution. Communication in Statistics-Theory and Methods 1975;4:469-481
21. Swain JJ, Venkatraman S, Wilson JR. Least-squares estimation of distribution functions in Johnson's translation system. Journal of Statistical Computation and Simulation 1988;29(4):271-297.
22. Tahir MH, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The logistic-X family of distributions and its applications. Communications in Statistics-Theory and Methods 2016;45(24):7326-7349.
23. Yahaya A, Alaku AY. On generalized Weibull-Rayleigh distribution: Its Properties and Applications. ATBU J. of Sci., Techn. and Edu 2018;5(1):217-231.
24. Yahaya A, Ieren TG. A note on the transmuted Weibull-Rayleigh distribution. Edited proceedings of 1st Int. Conf. of Nigeria Stat. Soc 2017;1:7-11.