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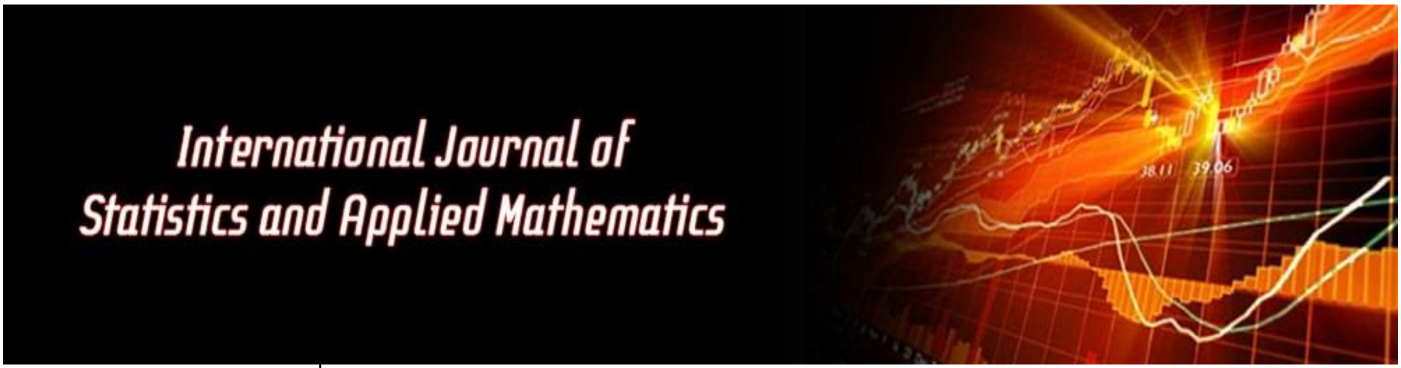

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Study of algebraic and holomorphic vector bundles

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Abstract
Let X be a 2-dimensional complex smooth algebraic variety and let X_{ho1} be the associated complex manifold. If X_{ho1} is not compact, usually X_{ho1} admits holomorphic vector bundles that are not algebrazable and non-isomorphic algebraic vector bundles which are isomorphic as holomorphic vector bundles. In this chapter we want to study some cases in which these phenomena of manifolds such that every (holomorphic or algebraic) vector bundle on them is an extension of line bundles. This is always true of dimension >1.

Keywords: Manifold, admits, holomorphic, algebrazable, affine, coherent, sheaf, irreducible, stein, triviality

1. Introduction

Let X be a complex, smooth, connected quasi-projective 2-dimensional algebraic variety. Let X_{ho1} be the associated analytic manifold. For any coherent algebraic sheaf, U, let U_{ho1} be the associated analytic coherent sheaf. We assume that X is a proper modification of an affine irreducible variety Y, i.e we assume the existence of an affine 2-dimensional normal variety Y and a proper subjective map f: X →Y. Since Y_{ho1} is a Stein variety, the map $f_{ho1:}$ $X_{ho1} \rightarrow Y_{ho1}$ induced by f shows that X_{ho1} is holomorphically 0-convex. We do not assume that Y is smooth. Let T be the union of all complete 1-dimensional algebraic sub varieties of X and set S: = f(T). Hence S is finite. We assume T ≠ ϴ, i.e X_{ho1} not Stein, because if T = ϴ our general result (Theorem 1.2.1) has as assumption the triviality of every holomorphic vector bundle on X_{ho1} whose restriction to a small bidisc is trivial, i.e it has as assumption the triviality of every holomorphic vector bundle on X_{ho1}. Our methods do not give any new result for Stein or affine varieties. The aim of this chapter is to proof following results 1.1.1 and 1.1.2.

1.1 Theorem

Let T be a smooth connected projective complex curve of genus q > 0. Let X: = V (L) be the total space of a line bundle L on T. Let us assume deg(L) < -q and h' (T,L*) = 0. Let u: X → T be the projection. Then every holomorphic vector bundle E on X is algebraizable and it is an extension of the line bundles of the form u*(U) with U ∈ Pic_{ho1} (T_{ho1}) ≅ Pic (T).

Taking Riemann-Roch as a particular case of 1.1.1 we have the following result. {Corollary 1.1.2

Let T be a smooth connected projective complex curve of genus q ≥ 0. Let X: = V(L) be the total space of a line bundle L on T. Let us assume deg (L) < min {0, 2 – 2q}. Let u: X → T be the projection. Then every holomorphic vector bundle E on X is algebraizable and it is an extension of line bundle of the form u*(U) with U ∈ Pic_{ho1} (T_{ho1}) ≅ Pic (T).

To prove these results we will prove a more general result below in which Y is not assumed to be a cone. An approximation theorem for holomorphic vector bundles on X_{ho1} by algebraic vector bundles will be delt in sub section (1.1.3).

1.1.1 The main results and the proof of

Set S: = f(t). Let I_T be the ideal sheaf of T in X and J the ideal sheaf of T_{ho1} in X_{ho1}. Let $T^{(n)}$ (resp. $T^{(n)}{}_{ho1}$), n ≥ 0, be the nth infinitesimal neighborhood of T in X (resp. of T_{ho1} in X_{ho1}) (hence with $I^{n+1}{}_T$ (resp. J^{n+1}) as ideal sheaf). Hence $T^{(0):}$ = T. Since X and X_{ho1} are smooth, T is a Cartier divisor of T and T_{ho1} is a Cartier division on X_{ho1}.

Hence the conormal sheaves I_T/I_T^2 and J/J^2 are line bundle on T and T_{ho1} and we have $(I_T/I_T^2)^{\otimes n} \cong S^n(I_T/ I_T^2) \cong (I_T^n/I_T^{(n+1)})$ and $(J/J^2)^{\otimes n} \cong S^n(J/J^2) \cong J^n/J^{(n+1)}$ for all integers n > 0. It is noted that by GAGA $Pic(T^{(n)}) \cong Pic_{ho1}(T_{ho1}{}^{(n)})$. Hence from the exact sequence on X:

$$0 \to I_T^{n+1} \to O_{T(n)} \to O_{T(n-1)} \to 0\ldots \quad (1.1)$$

And the corresponding exact sequence $(1.1)_{ho1}$ on X_{ho1} we obtain the following exact sequence in the Zariski topology:

$$0 \to S^n(I_T/ I_T^2) \to O_{T(n)} \to O_{T(n-1)} \to 0 \ldots \quad (1.2)$$

And the corresponding exact sequence $(1.2)_{ho1}$ on X_{ho1} in the Euclidean topology. Let T^∞ (resp. $T^\infty{}_{ho1}$) be the formal scheme (resp. formal analytic space) which is the limit of the schemes (resp. complex spaces) $T^{(n)}$ (resp. $T^\infty{}_{ho1}$). Hence $T^\infty{}_{ho1}$ is the completion of X_{ho1} along T_{ho1}. The purpose of this section is the proof of the following results.

1.2 Theorem

Let X be a smooth (non affine) quasi-projective complex surface such that there is a proper bi-rational morphism f: X → Y with Y affine. Let T ≠ ϴ be the union of all the complete algebraic curves contained in X. We assume that the following conditions are satisfied:

1. every holomorphic line bundle on X_{ho1} is algebraic;
2. the irreducible components of T are smooth:
3. for every irreducible component D of T and for a general P ≅ D there is a closed algebraic subvariety V of X, V ≅ A^1 such that V intersects transversally D and exactly at P;
4. $H^1(T, S^n(I_T/ I^2{}_T)) = 0$ for every n > 0:
5. There is a fundamental system of neighborhoods $\{U_n\}_n \cong$ N of T_{ho1} in X_{ho1} (in the Euclidean topology) such that every holomorphic rank 2 vector bundle M on X_{ho1} with M| Un trivial for some n is trivial.

Then every holomorphic vector bundle on X_{ho1} is algebraizable and is an extension of line bundles. We shall see that all these condition are satisfied in the case needed to prove Theorem 1.1.1 and hence Corollary 1.1.2.

1.2.1 Lemma

Let F be an algebraic vector bundle on X and F_{ho1} the associated holomorphic vector bundle on X_{ho1}. Then $H^1(X,F)$, $H^1(T^\infty, F/T^\infty)$, $H^1(T^\infty{}_{ho1}, F_{ho1}, \backslash T^\infty{}_{ho1})$ and $H^1(X_{ho1}, F_{ho1})$ are finite dimensional vector with the same dimension.

Proof

For all integers n > 0 we will use the tensorisazion with F(resp. F_{ho1} in the complex topology) of the exact sequence (1.2) (resp. $(1.2)_{ho1}$). Since F is locally free these sequences are exact. Since T is contracted by f, there is an integer m such that for all n> m we have $H^1(T^{(n)},F\otimes S^n(I_T/ I^2{}_T) = H^1(T^{(n)}{}_{ho1}, F_{ho1}\otimes S^n(I_T/ I^2{}_T) = 0$. Since dim(T) = 1, we have $h^2(X,G) = h^2(T_{ho1}, G_{ho1} \mid T_{ho1}) = 0$ for every algebraic co-homology groups $H^1(T^\infty, F|T)$ and $H(T^\infty{}_{ho1}, F_{ho1}, T_{ho1})$ are finite dimensional vector spaces with the same dimension. Since Y_{ho1} is Stein, Y is affine and f and f_{ho1} are proper, the Leray spectral sequence of f and the formal function Theorem for proper maps [6].

1.2.3 Lemma

Let A, B be algebraic vector bundles on X and A_{ho1}. Then $Ext^1(X; A,B)$, $Ext^1(T^\infty; A\backslash T^\infty, B/ T^\infty)$, $Ext^1(T^\infty{}_{ho1}; A/ T^\infty{}_{ho1}, B/ T^\infty{}_{ho1})$ and $Ext^1(X_{ho1}; A_{ho1}; B_{ho1})$ are finite dimensional vector spaces with the same dimension.

Proof

Since A and B are locally free, we have $Ext^1 \cong H'(X, Hom(A,B)$ and $Ext^1(X_{ho1}; A_{ho1}; B_{ho1}) \cong H^1(X_{ho1},Hom(A_{ho1}; B_{ho1}))$ and F:= Hom(A,B) is an algebraic vector bundle. Hence the result follows from Lemma 1.2.2.

1.2.4 Lemma

We assume $H^1(T,S^n(I_T/I^2{}_T)) = 0$ for every n>0. Let A (resp. B) be a formal vector bundle on T^∞(resp. $T^\infty{}_{ho1}$). If A|T (resp. B/T_{ho1}) is trivial, then A (resp. B) is trivial.

Proof.

We will prove the triviality of A, since same proof works for B Tensor the exact sequence (1.2) for n=1 with A. since $H^1(T, I_T/I^2{}_{T)} = 0$ and A|T is trivial we obtain that a base of $H^0(T, A|T)$ lifts to sections of $A|T^{(1)}$ Whose restriction to T are linearly independent. Hence these sections induce a trivialization of $A|T^{(1)}$. And so on using (1.2) for all n.

1.2.5 Lemma

We assume $H^1(T, S^n(I_T/I^2{}_T) = 0$ for every n > 0. Let A (resp. B) be an algebraic (resp. analytic) vector bundle of X (resp.X_{ho1}). If A|T (resp. B| T_{ho1}) is trivial, then there is a Zarisk (resp. Euclidean) neighbourhood U(resp. V) of T such that A|U (resp. B|V) is trivial.

Proof.

By Lemma 2.4 A| T^∞ and B| $T^\infty{}_{ho1}$ are trivial. It is sufficient to extend a base of $H^0(T^\infty, A|T)$ (resp.$H^0(T^\infty{}_{ho1}; B| T^\infty{}_{ho1})$ to a Zariski neighbourhood U(resp. Euclidean neighbourhood V) of T. Since S: = f (T) is finite, S has a fundamental system of neighbourhood in Y (resp. Y_{ho1}) formed by affine (resp. Stein) neighbourhoods. By Theorem A for affine varieties (resp. Stein spaces) the extension of the sections follows from the formal function Theorem, for proper maps for the algebraic case and [1] for the analytic case.

1.2.6 Lemma

Assume that every holomorphic vector bundle F on X_{ho1} with rank (F) ≤ 2 is algebrazable (resp. it is an extension of holomorphic (resp. algebraic) line bundles). Then every holomorphic vector bundle on X_{ho1} is algebrazable (resp. it is an extension of holomorphic (resp. algebraic) line bundles).

Proof.

Let H be a very ample line bundle on S, Let E be a vector bundle on X_{ho1} with r: rank (E) ≥ 3. We will use induction on r. Since $f_{ho1}(E)$ is a coherent sheaf and Y_{ho1} is Stein, $f_{ho1}(E)$ is generated by global sections. Noted that $f_{ho1}(E)$ is locally free outside S. Hence we may find a sufficiently general global section of $f_{ho1}(E)$ (i.e. of E) generating a sub-bundle of $f_{ho1}{}^*(E)$ outside a small neighbourhood of S. This general section generates a rank 1 sub sheaf R, R ≅ O_{xho1}, of E which is a sub bundle of E (i.e. with E/R locally free) outside a small neighbourhood of T_{ho1}. Since dim (T) = 1 and T is projective, we may twist E by a high power H⊗ m_{ho1}, m > 0, of H_{ho1} so that E ⊗ H⊗ $m_{ho1}$is generated by global sections in a Euclidean neighbourhood of T (Theorems A and B for projective morphisms between complex spaces [1], for a more general case), Taking E⊗ H⊗ m_{ho1} instead of E in the first part of the proof, we obtain that (H⊗ m_{ho1})* is a sub bundle of

E. Let E' be the rank r-1 quotient E / (H⊗ m_{ho1})*. We conclude by the inductive assumption on r and Lemma 2.3.

1.2.7 Proof of Theorem

According to Lemma 1.2.6, in order to check that every holomorphic vector E on X_{ho1} with rank (E)=2 is algeraizable. In addition, Lemma 2.3 says that to prove all the assertions of 1.2.1 it is sufficient to show that E is an extension (as holomorphic bundle) of two holomorphic line bundles. We fix an irreducible component D of T (hence smooth). If D is rational we fix a curve V ≅ A^1 such that V intersects D exactly at one point, p. If D has genus q > 0 we fix finitely many such curves, V(1),……V(s) with p(i): = D∩V(i) sufficiently general and such that there is an effective divisor $\sum 1_{\leq i \leq sni}$ P(i), n_1 ∊ N defining a large multiple of a very ample divisor H on D and a divisor $\sum 1_{\leq i \leq sni}$ P(i), m_1 ∊ N defining det (E | D)$^{\otimes}{}_m$ for some m > 0. This is possible with, say, s = 3q.Set V:=V(1) U…UV (s). Since every holomorphic vector bundle on a one- dimensional stein manifold is trivial [3], E | V is trivial and there are several subjective maps (as analytic coherent sheaves) E | V →Ovho1. We fix one of them, t, and set E':= Ker(t).E' is a vector bundle on X_{ho1} because V is a Cartier divisor of X_{ho1}[7]. It is noted that E | D is obtained from E | D making the elementary transformation [7] corresponding to t | D. Hence we see that there is a large integer m and a vector bundle E" on X_{ho1} such that we have an exact sequence

$$0 \rightarrow E" \rightarrow E\,(mH_{ho1}) \rightarrow M \rightarrow 0\ldots \qquad (1.3)$$

With M trivial rank 1 sheaf of O_{vho1}- modules and with E" | D trivial. We may do the same construction for all components of T with the same integer m and obtain an analytic vector bundle, E" on X_{ho1}, E" fitting in (1.3) with M structural sheaf of the union of finitely many divisors isomorphic to A*l* and with E" | D trivial. By Lemmas 1.2.3 and 1.2.4, E"|Un is trivial for some n. Hence E" is trivial by our assumption on Un. The existence of such E" and the fact the M is supported on an algebraic divisor, U' and it is a trivial line bundle on U gives the existence of R ∈ Pic_{ho1} (X_{ho1}) such that R is a sub bundle of E, i.e. by E/R ∈ Pic_{ho1} (X_{ho1}). Hence E is an extension of two line bundles, as wanted.

1.2.8 Proof of Theorem

Now we will check that X satisfies all the assumptions of 1.2.1. It is noted that $Pic_{ho1}(X_{ho1})$ ≅ $u^*_{ho1}(Pic_{ho1}(T_{ho1})$ ≅ u*(Pic_{ho1}(T) (e.g. use the exponential sequence and the fact that X is homotopic to T). Let T&X be the 0- section. Also noted that T is contractible in the algebraic category (say by the morphism f; X→Y) and the corresponding contraction y is an affine variety. Let U be any neighbourhood of T in X_{ho1} in the Euclidean topology and let G be a holomorphic vector bundle on X_{ho1} with G|U trivial. Since Y is normal by definition of contraction, we have $f^*_{ho1}(OX_{ho1}= OY_{ho1})$. This equality and the triviality of G|U imply that $f^*_{ho1}(G)$ ≅ G. Since Y is contractible, G' is trivial as topological vector bundle. Since Y is Stein, by a theorem of Grauert (Oka's principal [4]) G' is trivial. Hence G is trivial and we have checked the 1st assumption of 1.2.1 needed to apply 1.2.1

1.3 An approximation result

Here is this short section we give the following easy approximation theorem for holomorphic vector bundles on X_{ho1} by a sequence of algebraic vector bundles.

1.3.1 Theorem

Let X be a normal 2- dimensional complex variety (X not affine) and such that there is a proper birational morphism f: X → Y affine. We assume that every holomorphic line bundle on X_{ho1} is algebraic. We fix an increasing sequence {U_n} n ≅ N of open subsets of X_{ho1} (in the Euclidean topology) which are relatively compact in X_{ho1} and with $U_{n≅}$n U_n = X_{ho1}. We fix a holomorphic vector bundle F on x. Then there is a sequence of algebraic vector bundles {En} $_{n≅N}$ such that (En) $_{ho1}$ | ≅ F | Un (as holomorphic vector bundles) for every n ≅ N.

Proof.

We many assume X connected and hence we may assume that F has constant rank, r. Since the case r = 1 is assumed to be true, we may assume r > 1. Let T be the exceptional set of f (hence T union of the 1-dimensional compact sub varieties of X_{ho1}, if any, and S: = f(T) is finite). First we assume r = 2. Using Theorem A for the coherent sheaf f_{ho1}(F) and the fact that f is an isomorphism outside a set with finite image in Y, we obtain the existence of a global section of F which generates a sub bundle of F outside a Euclidean neighborhood V of T and outside a discrete set Z of X_{ho1} with Z ∩ Sing(X) = ∅; the last condition is easily satisfied because Sing(X) is finite and we just need to take a section of F not vanishing at any point of Sing(X). We claim that, as in the proof of Lemma 1.2.6, we may find M≅ $Pic_{ho1}(X_{ho1})$ and L ≅ $Pic_{ho1}(X_{ho1})$ such that F fits in the exact sequence on X_{ho1}

$$0 \rightarrow M \rightarrow F \rightarrow L{\otimes}I_z \rightarrow 0\ \ldots \qquad (1.4)$$

The only difference with respect to the proof of 1.2.6 is that in (1.4) we take Z with its reduced structure. This is a kind of Bertini's theorem, but it is not essential. If Z is just a 0-dimensional uncreduced subspace of X_{ho1} the proof below will work with just notational modifications. We set Z(n) = Z ∩ Un. Hence each Z(n) is a finite 0-dimensional sub scheme of X because Z is discrete and U_n relatively compact in X_{ho1}. For each n < 0 we consider the set of all extensions (with M, L and X(n)) fixed

$$0 \rightarrow M \rightarrow A(n) \rightarrow L{\otimes}I_{z\,|\,n\,|} \rightarrow 0\ldots \qquad (1.5)$$

both in the analytic and in the algebraic category. By Lemma 1.2.3 if Z | n | = ∅ this set of extensions is a finite dimensional vector space and gives the same objects in both categories. We assume Z | n | ≠ ∅, We use that Z is locally a complete intersection (this being true if Z is not reduced) and contained in the smooth part of X. We use the local to global spectral sequence of the Ext- functors and the fact that H^2(X, F)= $H^2(X_{ho1}$, G)= 0 for all coherent sheaves F and G. As in [2] (which corrects the corresponding calculations and statements given in [4] we obtain that the vector space Ext^1(X; L⊗$I_{z\,|\,n\,|}$, M) is the direct sum of H^1(X, M ⊗ L*) and a finite dimensional vector space W(n) with dim (W(n) = card (Z(n)red) and that the same is true for the extensions as analytic sheaves. Furthermore, the proof of Lemma 1.2.2 shows that if T& U_n (which is the case for large n we have Ext^1(Un; L⊗ $I_{z\,|\,n\,|}$, M)≅ W(n) ⊗ $Ext^1(X_{ho1}$; L,M) ≅ W(n) ⊗ Ext^1(X; L,M) ≅ Ext^1(X, L⊗ $I_{z\,|\,n\,|}$, M) (the algebraic extension). Hence for every such large nF | U_n induces an extension e[n] as in (5).this extension is algebrazable, i.e. it gives a coherent algebraic sheaf E_n which is locally free outside Z(n). Furthermore in [2] (correcting [5] it is stated and proved the necessary and sufficient conditions for the local freeness of A(n) (Cayley-Bacharch condition) This condition is satisfied for the extension e[n] because F is locally free. Hence E_n is locally free and by construction $(E_n)_{ho1}$ | ≅ F | U_n

for large n. Taking interated extensions we obtain the case r > 2 as in the proof of Lemma 1.2.6.

2. Conclusion

Let X be smooth complex algebraic surface such that there is a proper bit rational morphism f: S → Y with Y an affine variety. Let X_{ho1} be the 2-dimensional complex manifold associated to X. Here we give conditions on X which imply that every holomorphic vector bundle on X is algebraizable and it is an extension of line bundles. In this chapter we want to study some cases in which these phenomena do not occur. Furthermore, we will give interesting examples of manifolds such that every (holomorphic or algebraic) vector bundle on them is an extension of line bundles. This is always true for smooth compact curves, but false one very projective variety of dimension > 1. We also give an approximation theorem of holomorphic vector bundles on X_{ho1} (X normal algebraic surface) by algebraic vector bundles.

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