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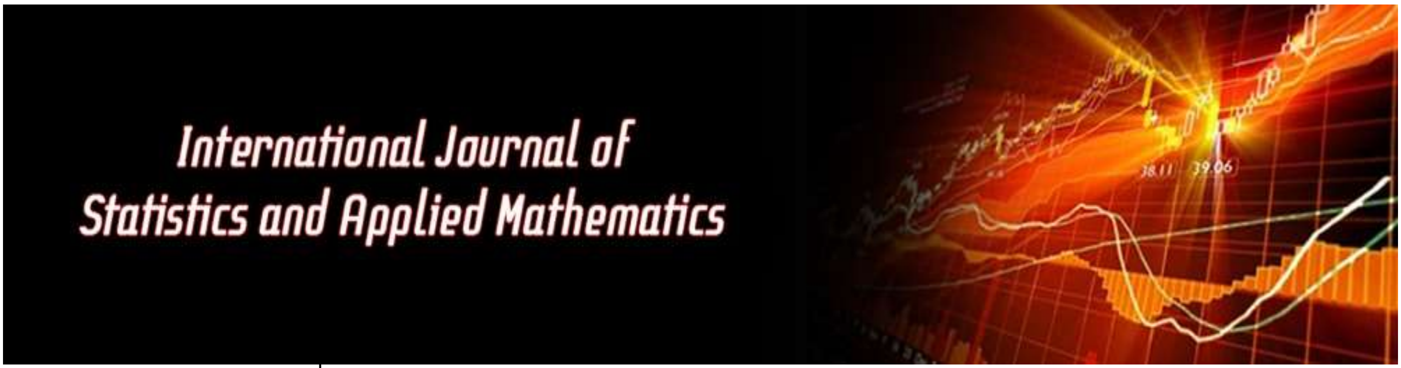

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On the existence and uniqueness of solutions to coupled system of burgers equations

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Abstract
The study seeks to prove the existence and uniqueness of solution of a coupled system of Burger's equations using Energy method. The solution and corresponding derivatives are bounded on the defined domain and we were able to prove the convergence of the solution to a solution and also to establish the uniqueness of the solution.

Keywords: Existence and uniqueness, coupled system of burger's equations, energy estimates

1. Introduction
Burger's equation is a simplified form of Navier Stokes Equation. In 1915, Harry Bateman [1] (1882-1946) modeled the initial and boundary value problem:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}, \quad 0<x<L, \quad 0<t<\tau \tag{1.1}$$

$$\left.\begin{aligned} &u(x,0)=u_0(x), \quad 0<x<L \\ &u(0,t)=f_1(t), \quad u(L,t)=f_2(t), \quad 0<t<\tau \end{aligned}\right\} \tag{1.2}$$

where u, x, t and V are the velocity, spatial coordinate, time and kinematic viscosity respectively.

$u_0, f_1 \& f_2$ are prescribed functions of variables depending upon the specific conditions for the problem. The equation (1.1) is known as Burgers' equation acknowledging the contribution of Johannes Martinus Burgers [2] (1895-1981) who used it to describe the mathematical modeling of turbulence. Since the appearance of this problem, many researchers have worked on it in many fields of applications.

2. Formal Solution
In this paper we seek to prove the existence and uniqueness of solution of one dimensional coupled system of Burger's equation (1)-(2) using Energy inequalities method

$$\begin{aligned} &\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}+\alpha(u-v)=0(i) \\ &\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}+v\frac{\partial v}{\partial x}+\beta(v-u)=0(ii) \end{aligned} \tag{2.1}$$

with initial and boundary conditions

$$\left.\begin{aligned} &u(x,0)=u_0,\ v(x,0)=v_0 \\ &u(0,t)=v(0,t)=0=u(2\pi,t)=v(2\pi,t) \end{aligned}\right\} \tag{2.2}$$

With α,β constant u_0, v_0 real valued functions.

We set first part of our problem in the form of Theorem for the existence of solution.

Theorem 1

If $\frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 v_0}{\partial x^2} \in L_2(0,2\pi)$ for any $T>0\left(t\in(0,T)\right)$, there exists classical solution to coupled system of Burgers equation (2.1)-(2.22).

To prove this Theorem we shall require the following three steps:

1. Derivation of energy estimates. Here we determine upper bounds on L_2 norms of u,v and their higher derivatives. This process is called a priori energy estimates,
2. Construction of approximate solutions. Here we seek the recourse of ODE theory to get local in time solution to a finite dimensional approximations u_n, v_n of (2.1)-(2.2) satisfying same energy bounds as u,v of the energy estimates,
3. Convergence of approximate solution to a solution. We use compactness argument to show that there exists a subsequence of $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ that converges and that the limiting solution satisfies (2.1)-(2.2).

The following inequalities and lemma will be used in the derivation of a priori Energy estimates

(a) Gronwall's Inequality [3]

Assume $k(t), g(t) \in C^0 \,\&\, a(t)\in C^1$ for $t\geq 0$ and it satisfies $a'(t)\leq k(t)a(t)+g(t)$

Then

$$a(t)\leq \mu(t)a(0)+\int \frac{\mu(t)}{\mu(\tau)}g(\tau)d\tau, \quad \text{where} \quad \mu(t)\exp\left[\int_0^t k(\tau)d\tau\right]$$

(b) Young's inequality [4]

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}, \quad Let \quad p=q=2$$

(c) Cauchy Schwarz inequality [5]

$$\left|\langle x,y\rangle\right| \leq \|x\|\|y\|$$

Lemma1: The following statements hold:

1. If $\frac{\partial v}{\partial x}\in L_2(0,1)$ and v satisfies $v(0)=0=v(2\pi)$, then $\|v\|\leq\|v\|_\infty\leq\left\|\frac{\partial v}{\partial x}\right\|$.
2. If $\frac{\partial^2 v}{\partial x^2}\in L_2(0,2\pi)$ and v satisfies $v(0)=0=v(1)$, then $\left\|\frac{\partial v}{\partial x}\right\|\leq\left\|\frac{\partial v}{\partial x}\right\|_\infty\leq\left\|\frac{\partial^2 v}{\partial x^2}\right\|$.

Proof

1. A Priori Energy Estimate

For a priori energy estimate we start of by taking the inner product of (2.1) (*i*) with u

$$\left\langle u,\frac{\partial u}{\partial t}\right\rangle-\left\langle u,\frac{\partial^2 u}{\partial x^2}\right\rangle+\left\langle u,u\frac{\partial u}{\partial x}\right\rangle+\alpha\langle u,(u-v)\rangle=0 \tag{2.3}$$

Integrating over $\Omega=(0,2\pi)$ by applying the divergence theorem and using the boundary conditions (2.2) on the boundary $\partial\Omega$ we have [5]

$$\int_\Omega u\frac{\partial u}{\partial t}dx-\int_\Omega u\frac{\partial^2 u}{\partial x^2}dx+\int_\Omega u^2\frac{\partial u}{\partial x}dx+\alpha\int_\Omega u(u-v)dx=0 \tag{2.4}$$

Then (2.4) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2+\|\nabla u\|_{L_2}^2-\alpha\int_\Omega u(u-v)dx=0 \tag{2.5}$$

Then using Cauchy Schwarz inequality and Young's inequality we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2+\|\nabla u\|_{L_2}^2\le\frac{\alpha}{2}\left(\|u\|_{L_2}^2+\|v\|_{L_2}^2\right) \tag{2.6}$$

Since the two equations in (2.1) are similar we have the results of (2.1) (*ii*) as

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L_2}^2+\|\nabla v\|_{L_2}^2\le\frac{\beta}{2}\left(\|v\|_{L_2}^2+\|u\|_{L_2}^2\right) \tag{2.7}$$

Combining (2.6) and (2.7) we get

$$\frac{d}{dt}\|(u,v)\|_{L_2}^2+\|\nabla(u,v)\|_{L_2}^2\le\rho\|(u,v)\|_{L_2}^2 \tag{2.8}$$

Dropping the second term of the LHS and combining the two terms of the RHS we get

$$\frac{d}{dt}\|(u,v)\|_{L_2}^2\le\rho\|(u,v)\|_{L_2}^2 \tag{2.9}$$

By *Gronwall's inequality:*

$$\|(u,v)\|_{L_2}^2\le\|(u_0,v_0)\|_{L_2}^2\exp\left(\int_0^t\rho d\tau\right)=E_0 \tag{2.10}$$

Now doing the time integration of (2.8) we have

$$\|(u,v)\|_{L_2}^2+\kappa\int_0^t\|\nabla(u,v)\|_{L_2}^2\,d\tau\le\|(u_0,v_0)\|_{L_2}^2+\rho\int_0^t\|(u,v)\|_{L_2}^2\,d\tau \tag{2.11}$$

Dropping $\|(u,v)\|_{L_2}^2$ on the L.H.S of (2.11) and using (2.10) to estimate R.H.S of (2.11) we obtain for

$$\kappa>0,\int_0^t\|\nabla(u,v)\|_{L_2}^2\,d\tau\le c_1\|(u_0,v_0)\|_{L_2}^2+c_2\|(u_0,v_0)\|_{L_2}^2\int_0^t\exp(\rho\tau)d\tau\le c\|(u_0,v_0)\|_{L_2}^2\int_0^t\exp(\rho\tau)d\tau=E_0 \tag{2.12}$$

We have shown that $\|(u,v)\|_{L_2}^2$ and $\int_0^t\|\nabla(u,v)\|_{L_2}^2\,d\tau$ are bounded if the solution exists to the system of coupled Burgers' equation (2.1)-(2.2). Next we determine the regularity of the solution by taking the x derivative in the equation (2.1) and we have

$$\begin{aligned}&\frac{\partial}{\partial t}(\nabla u)-\frac{\partial^2}{\partial x^2}(\nabla u)+\frac{\partial}{\partial x}(u\nabla u)+\alpha\frac{\partial}{\partial x}(u-v)=0(i)\\&\frac{\partial}{\partial t}(\nabla v)-\frac{\partial^2}{\partial x^2}(\nabla v)+\frac{\partial}{\partial x}(v\nabla v)+\beta\frac{\partial}{\partial x}(u-v)=0(ii)\end{aligned} \tag{2.13}$$

Taking the inner product (2.13) (*i*) with ∇u, using $\Delta u=0; x=(0,2\pi)$, and integrating over $\Omega=(0,2\pi)$ by applying the divergence theorem and using the boundary conditions on $\partial\Omega$ we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\nabla u)^2dx-\int_\Omega(\Delta u)^2dx=-\int_\Omega u\nabla u\Delta u dx-\alpha\int_\Omega\nabla u(\nabla u-\nabla v)dx \tag{2.14}$$

On using the Cauchy Schwarz inequality and Young's inequality we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L_2}^2+\frac{1}{2}\|\Delta u\|_{L_2}^2\le\eta(t)\|\nabla u\|_{L_2}^2+\frac{\alpha}{2}\|\nabla v\|_{L_2}^2 \tag{2.15}$$

Similarly from (2.13) (ii) we get $\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L_2}^2+\frac{1}{2}\|\Delta v\|_{L_2}^2\le\sigma(t)\|\nabla v\|_{L_2}^2+\frac{\beta}{2}\|\nabla u\|_{L_2}^2$ (2.16)

Then combining (2.15) and (2.16) we get $\frac{d}{dt}\left\|\nabla(u,v)\right\|_{L_2}^2+\left\|\Delta(u,v)\right\|_{L_2}^2\le\lambda(t)\left\|\nabla(u,v)\right\|_{L_2}^2$ (2.17)

Dropping the second term of the LHS and adding the two terms of the RHS we get

$$\frac{d}{dt}\left\|\nabla(u,v)\right\|_{L_2}^2\le\lambda(t)\left\|\nabla(u,v)\right\|_{L_2}^2 \tag{2.18}$$

On using *Gronwall's Inequality* we get

$$\left\|\nabla(u,v)\right\|_{L_2}^2\le\left\|\nabla(u_0,v_0)\right\|_{L_2}^2\exp\int_0^t\lambda(\tau)\,d\tau\le E_1 \tag{2.19}$$

Now going back to (2.17) and carrying out time integration and applying *Gronwall's Inequality*

$$\left\|\nabla(u,v)\right\|_{L_2}^2+\int_0^t\left\|\Delta(u,v)\right\|_{L_2}^2\,d\tau\le\left\|\nabla(u_0,v_0)\right\|_{L_2}^2\exp\int_0^t\lambda(\tau)d\tau\le E_1 \tag{2.20}$$

Dropping the first term and using (2.19) we get

$$\int_0^t\left\|\Delta(u,v)\right\|_{L_2}^2\,d\tau\le\left\|\nabla(u_0,v_0)\right\|_{L_2}^2\exp\left[\int_0^t\lambda(\tau)\,d\tau\right]\le E_1 \tag{2.21}$$

We have shown that $\left\|\nabla(u,v)\right\|_{L_2}^2$ and $\int_0^t\left\|\Delta(u,v)\right\|_{L_2}^2\,d\tau$ are bounded if the solution exists to the system of coupled Burger's equation (2.1)-(2.2). We take the χ derivative of the system of equation (2.13) and we have

$$\begin{aligned}&\frac{\partial}{\partial t}(\Delta u)-\frac{\partial^3}{\partial x^3}(\nabla u)+\frac{\partial^2}{\partial x^2}(u\nabla u)+\alpha\frac{\partial^2}{\partial x^2}(u-v)=0(i)\\&\frac{\partial}{\partial t}(\Delta v)-\frac{\partial^3}{\partial x^3}(\nabla v)+\frac{\partial^2}{\partial x^2}(v\nabla v)+\beta\frac{\partial^2}{\partial x^2}(u-v)=0(ii)\end{aligned} \tag{2.22}$$

Taking the inner product (2.22) (*i*) with Δu , using $\Delta u=0; x=(0,2\pi)$ and integrating over $\Omega=(0,2\pi)$ by applying the divergence theorem and using the boundary conditions on $\partial\Omega$ we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\Delta u)^2dx-\int_\Omega\left(\frac{\partial^3u}{\partial x^3}\right)^2dx+\int_\Omega\Delta u\Delta(u\nabla u)\,dx+\alpha\int_\Omega\Delta u\frac{\partial^2}{\partial x^2}(u-v)\,dx=0 \tag{2.23}$$

On integrating by parts and using the Cauchy Schwarz inequality and Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\Delta u\right\|_{L_2}^2+\left\|\frac{\partial^3u}{\partial x^3}\right\|_{L_2}^2\le\frac{1}{2}\left\|\frac{\partial^3u}{\partial x^3}\right\|_2^2+\frac{25}{2}k_1(t)\left\|\Delta u\right\|_2^2+\alpha\left\|\Delta u\right\|_{L_2}^2+\frac{\alpha}{2}\left[\left\|\Delta u\right\|_{L_2}^2+\left\|\Delta v\right\|_{L_2}^2\right] \tag{2.24}$$

The RHS simplifies to:

$$\frac{1}{2}\frac{d}{dt}\left\|\Delta u\right\|_{L_2}^2+\frac{1}{2}\left\|\frac{\partial^3u}{\partial x^3}\right\|_{L_2}^2\le\left(\frac{25}{2}k_1(t)+\psi\right)\left\|\Delta u\right\|_2^2+\frac{\alpha}{2}\left\|\Delta v\right\|_{L_2}^2 \tag{2.25}$$

Similarly from (2.22) (ii) we get

$$\frac{1}{2}\frac{d}{dt}\left\|\Delta v\right\|_{L_2}^2+\frac{1}{2}\left\|\frac{\partial^3v}{\partial x^3}\right\|_{L_2}^2\le\left(\frac{25}{2}k_2(t)+\xi\right)\left\|\Delta v\right\|_2^2+\frac{\beta}{2}\left\|\Delta u\right\|_{L_2}^2 \tag{2.26}$$

Combining (2.25) and (2.26) we get

$$\frac{d}{dt}\left\|\Delta(u,v)\right\|_{L_2}^2+\left\|\frac{\partial^3}{\partial x^3}(u,v)\right\|_{L_2}^2\le(25k(t)+\rho)\left\|\Delta(u,v)\right\|_2^2 \tag{2.27}$$

Dropping the second term of the LHS and letting $(25k(t)+\rho)=\kappa(t)$ we get

$$\frac{d}{dt}\|\Delta(u,v)\|_{L_2}^2 \le \kappa(t)\|\Delta(u,v)\|_2^2 \tag{2.28}$$

On using *Gronwall's Inequality* we get,

$$\|\Delta(u,v)\|_{L_2}^2 \le \|\Delta(u_0,v_0)\|_{L_2}^2 \exp\int_0^t \kappa(\tau)d\tau \le E_2 \tag{2.29}$$

Now going back to (2.27) and carrying out time integration

$$\|\Delta(u,v)\|_{L_2}^2 + \int_0^t \left\|\frac{\partial^3}{\partial x^3}(u,v)\right\|_{L_2}^2 d\tau \le \|\Delta(u_0,v_0)\|_{L_2}^2 + \int_0^t \kappa(t)\|\Delta(u,v)\|_2^2 d\tau \tag{2.30}$$

Dropping the first term and using (2.30) we get

$$\int_0^t \left\|\frac{\partial^3}{\partial x^3}(u,v)\right\|_{L_2}^2 d\tau \le \int_0^t \kappa(t)\|\Delta(u,v)\|_2^2 d\tau \le \|\Delta(u_0,v_0)\|_{L_2}^2 \exp\int_0^t k(\tau)d\tau \le E_2 \tag{2.31}$$

We have shown that $\|\Delta(u,v)\|_{L_2}^2$ and $\int_0^t \left\|\frac{\partial^3(u,v)}{\partial x^3}\right\|_{L_2}^2 d\tau$ are bounded if the solution exists to the system of coupled Burger's equation (2.1)-(2.2).

3. Galerkin's approximation

The following definitions will be of essence when we discuss the Galerkin's approximation method

Projection Operator: Let X be a vector space over K. A linear operator P on X is called a projection operator if $P^2 = P$.The projection operator P satisfies the properties $\langle P(x), y\rangle = \langle P(x), P(y)\rangle = \langle x, P(y)\rangle$ & $\|P\| = 1$ where $\langle .,.\rangle$ denotes the inner product of square integrable functions.

Galerkin method: This is a method in which we approximate the solution of a PDE by the projection of the solution and the solution into finite dimensional subspaces (Hunter, 2014) [6].

Parsevals identity or equation: Let E be an inner product space. Then $S = \{u_\alpha\}_{\alpha\in\Delta}$ is a basis for E if and only if for arbitrary

$x \in E$, $\sum_{\alpha\in\Delta}|\langle x, u_\alpha\rangle|^2 = \|x\|^2$, where $x = \sum_{\alpha\in\Delta}\langle x, u_\alpha\rangle u_\alpha$

The identity above is called Parsevals identity or equation.

Using Galerkin approximation method we start off by constructing finite approximation to (1)-(2). Let $\lambda_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}, i = \sqrt{-1}$, $\{\lambda_k(x)\}_{k=1}^\infty$ forms an orthonormal basis for functions on $L_2[0,2\pi]$ Let us define $E_k \in L_2[0,2\pi]$ a finite dimensional subspace $E_k = \left\{u \in L_2[0,2\pi] : u(x) = \sum_{k=1}^n c_k\lambda_k(x)\right\}$, where $c_k = \langle u, \lambda_k\rangle_{L_2}$ & $u \in L_2(0,2\pi)$ if and only if $\sum_{k\in N}|c_k|^2 < \infty$.

We denote by $P_N : L_2(0,2\pi) \to E_N \in L_2(0,2\pi)$ [6]

Orthogonal projection onto E_N defined by $P_k\left(\sum_{k\in N}c_k\lambda_k\right) = \sum_{k=1}^N c_k\lambda_k$.

The following Lemmas are essential in the sequel. We state them without proof

Lemma 2: For any integer $K \ge 0$; if $u \in E_N$ then for any $\lambda \in L_2(0,2\pi)$ then we have $\langle \nabla_x^k u, \nabla_x^k P_k\lambda\rangle = \langle \nabla_x^k u, \nabla_x^k \lambda\rangle$.

Lemma 3: If $\lambda_k \to \lambda$ in $C'[0,2\pi]$ then $\|P_k\lambda_k - \lambda\|_\infty \to 0$ as $k \to \infty$

Define Galerkin approximation to (1) by seeking $u_N \in E_N$ with finite representation

$u_n(x,t)=\sum_{k=1}^{n}a_k(t)\lambda_k(x)$, $v_n(x,t)=\sum_{k=1}^{n}b_k(t)\lambda_k(x)$

Then a_k and b_k are chosen to satisfy

$$\frac{\partial}{\partial t}(u_n)-\nabla^2 u_n=-P_N\left(u_n\nabla u_n+\alpha(u_n-v_n)\right),\frac{\partial}{\partial t}(v_n)-\nabla^2 v_n=-P_N\left(v_n\nabla v_n+\beta\nabla(u_n-v_n)\right) \quad (2.32)$$
$$u_n(x,0)=P_N u_0,\mathrm{v}_n(x,0)=P_N v_0$$

Which is equivalent to the set of finite nonlinear ODES for $K=1,...,n.$

$$\frac{d}{dt}(a_k)-k^2a_k=i\left(\lambda_k\sum_{m=1}^{n}\sum_{j=1}^{n}\left\{a_m(t)a_j(t)\lambda_{m+j}+a_m(t)a_j(t)\lambda_{m-j}\right\}+\alpha\sum_{m=1}^{n}\sum_{j=1}^{n}\left(a_m(t)\lambda_m(x)-a_j(t)\lambda_j(x)\right)\right)$$
$$\frac{d}{dt}(b_k)-k^2b_k=i\left(\lambda_k\sum_{m=1}^{n}\sum_{j=1}^{n}\left\{b_m(t)b_j(t)\lambda_{m+j}+b_m(t)b_j(t)\lambda_{m-j}\right\}+\beta\sum_{m=1}^{n}\sum_{j=1}^{n}\left(b_m(t)\lambda_m(x)-b_j(t)\lambda_j(x)\right)\right) \quad (2.33)$$
$$a_k(0)=\left\langle\lambda_{k,}u_0\right\rangle,\mathrm{b}_k(0)=\left\langle\lambda_{k,}v_0\right\rangle$$

Since the equations in (2.1) are similar we work with one that is (2.1) (*i*) which corresponds to the projection equations (2.32) and generalize on the other (2.1) (*ii*) and the corresponding equations. We may write (2.32) in the integral form as

$$u_n(x,t)=P_nu_0+\int_0^t\left\{\nabla^2u_n-P_n\left[u_n\nabla u_n+\alpha(u_n-v_n)\right]\right\}d\tau \quad (2.34)$$
$$v_n(x,t)=P_nv_0+\int_0^t\left\{\nabla^2v_n-P_n\left[v_n\nabla v_n+\alpha(v_n-v_n)\right]\right\}d\tau$$

Proposition[7] *1*: For any integer n with $T>0$ there exists a unique solution to the ODE system (2.33) above for $t\in(0,T)$ implying unique solution u_n satisfying (2.32)

Proof

Taking the inner product of first equation of (2.32) with u_n

$$\left\langle u_n,\frac{\partial}{\partial_t}u_n\right\rangle-\left\langle u_n,\Delta u_n\right\rangle=-\left\langle u_n,P_n\left(u_n\nabla u_n+\alpha(u_n-v_n)\right)\right\rangle \quad (2.35)$$

And following the procedure for equations (2.3) to (2.6) we get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L_2}^2+\|\nabla u_n\|_{L_2}^2\le\frac{\alpha}{2}\left[\|u_n\|_{L_2}^2+\|v_n\|_{L_2}^2\right] \quad (2.36)$$

Similarly $\frac{1}{2}\frac{d}{dt}\|v_n\|_{L_2}^2+\|\nabla v_n\|_{L_2}^2\le\frac{\beta}{2}\left[\|v_n\|_{L_2}^2+\|u_n\|_{L_2}^2\right]$ (2.37)

On combining (2.36) and (2.37) $\frac{d}{dt}\|(u_n,v_n)\|_{L_2}^2+\kappa\|\nabla(u_n,v_n)\|_{L_2}^2\le\delta\|(u_n,v_n)\|_{L_2}^2$ (2.38)

Dropping the second term of LHS we get $\frac{d}{dt}\|(u_n,v_n)\|_{L_2}^2\le\delta\|(u_n,v_n)\|_{L_2}^2$ (2.39)

And using Gronwall's inequality, it follows that $\|(u_n,v_n)\|_{L_2}^2\le\|(u_0,v_0)\|_{L_2}^2\exp(\delta T)=E_0$ (2.40)

Where we take the maximum of $\|(u_n,v_n)\|\le E_0$ on $t\in(0,T)$

Thus from Parsevals' inequality $\sum_{k=1}^{n}a_k^2\le E_0$

Proposition[7] *2*: For any integer n the solution u_n found in Proposition (1) satisfies the same bounds as the a priori bounds on, for $j=0,1..$, $\left\|\frac{\partial^j u_n}{\partial x^j}\right\| \le C$, where C is independent of n and T and only depends on L_2 bounds of u_0''. Furthermore $\int_0^t \left\|\frac{\partial^j u_n}{\partial x^j}\right\|^2 \le C$, for $j=0,1..$, where C is independent of n and T, but only depends on L_2 bounds of u_0''

Proposition3 [7]: For any integer $n \ge 1$, the solution u_n satisfies above Proposition1.

Proof

We note that $\frac{\partial u_n}{\partial t} = \Delta u_n - P_n\left[u_n \nabla u_n + \alpha\left(u_n - v_n\right)\right]$ (2.41)

Taking the L_2 norm with $\|P\|_{L_2} = 1$ then we have $\left\|\frac{\partial u_n}{\partial t}\right\|_{L_2} \le \|\Delta u_n\|_{L_2} + \|u_n\|_{L_2}\|\nabla u_n\|_{L_2} + \alpha\|(u_n - v_n)\|_{L_2}$

Applying Lemma1 we get $\left\|\frac{\partial u_n}{\partial t}\right\|_{L_2} \le \|\Delta u_n\|_{L_2} + 2\|u_n\|_\infty \|\nabla u_n\|_{L_2} + \alpha\|(u_n - v_n)\|_{L_2} \le C$ (2.42)

Taking the x derivative of (2.41) $\frac{\partial}{\partial t}(\nabla u_n) = \frac{\partial^3 u_n}{\partial x^3} - P_n \frac{\partial}{\partial x}\left[u_n \nabla u_n + \alpha\left(u_n - v_n\right)\right]$ (2.43)

Since P_n and $\frac{\partial}{\partial x}$ do commute and $\|P_n\| = 1$, on taking L_2 estimate we get

$$\left\|\frac{\partial}{\partial t}\nabla u_n\right\|_{L_2} \le \left\|\frac{\partial^3 u_n}{\partial x^3}\right\|_{L_2} - \left\|\nabla u_n \nabla u_n + u_n \Delta u_n + \alpha \nabla\left(u_n - v_n\right)\right\|_{L_2} \tag{2.44}$$

We have from Lemma1

$$\|u_n\|_{L_2} \le \|u_n\|_\infty \ \& \left\|\frac{\partial u_n}{\partial x}\right\|_{L_2} \le \left\|\frac{\partial u_n}{\partial x}\right\|_\infty \left\|\frac{\partial}{\partial t}(\nabla u_n)\right\|_{L_2} \le \left\|\frac{\partial^3 u_n}{\partial x^3}\right\|_{L_2} + \left(\|\nabla u_n\|_\infty \|\nabla u_n\|_{L_2} + \|u_n\|_\infty \|\Delta u_n\|_{L_2}\right) + \alpha\left\|\nabla\left(u_n - v_n\right)\right\|_{L_2} \le C \tag{2.45}$$

Taking the x derivative of (2.43), we also note that (Simplifying using product rule)

$$\frac{\partial}{\partial t}\Delta u_n = \frac{\partial^4 u_n}{\partial x^4} - P_n \frac{\partial^2}{\partial x^2}\left[u_n \nabla u_n + \alpha \nabla\left(u_n - v_n\right)\right] \ \frac{\partial}{\partial t}\Delta u_n = \frac{\partial^4 u_n}{\partial x^4} - P_n\left[3\nabla u_n \Delta u_n + u_n \frac{\partial^3 u_n}{\partial x^3} + \alpha \Delta\left(u_n - v_n\right)\right] \tag{2.46}$$

Thus $\left\|\frac{\partial}{\partial t}\Delta u_n\right\|_{L_2} \le \left\|\frac{\partial^4 u_n}{\partial x^4}\right\|_{L_2} + 3\|\nabla u_n\|_\infty \|\Delta u_n\|_{L_2} + \|u_n\|_\infty \left\|\frac{\partial^3 u_n}{\partial x^3}\right\|_{L_2} + |\alpha|\|\Delta u_n\| - |\alpha|\|\Delta v_n\| \le C + \left\|\frac{\partial^4 u_n}{\partial x^4}\right\|_{L_2}$ (2.47)

Now taking t derivative of equation (2.41) we write

$$\frac{\partial^2 u_n}{\partial t^2} = \frac{\partial}{\partial t}\Delta u_n - \frac{\partial}{\partial t}P_n\left[u_n \nabla u_n + \alpha\left(u_n - v_n\right)\right] \ \frac{\partial^2 u_n}{\partial t^2} = \frac{\partial}{\partial t}\Delta u_n - P_n\left[u_n \frac{\partial}{\partial t}\nabla u_n + \frac{\partial}{\partial t}u_n \nabla u_n + \alpha\left(\frac{\partial}{\partial t}u_n - \frac{\partial}{\partial t}v_n\right)\right] \tag{2.48}$$

Then taking the L_2 norm and using Lemma1, it follows that

$$\left\|\frac{\partial^2 u_n}{\partial t^2}\right\| = \left\|\frac{\partial}{\partial t}\Delta u_n\right\| + \|P_n\|\left[\|u_n\|\left\|\frac{\partial}{\partial t}\nabla u_n\right\| + \left\|\frac{\partial}{\partial t}u_n\right\|\|\nabla u_n\| + |\alpha|\left\|\frac{\partial}{\partial t}u_n\right\| + |\alpha|\left\|\frac{\partial}{\partial t}v_n\right\|\right]$$

$$\left\|\frac{\partial^2 u_n}{\partial t^2}\right\| = \left\|\frac{\partial}{\partial t}\Delta u_n\right\| + \|u_n\|_\infty \left\|\frac{\partial}{\partial t}\nabla u_n\right\| + \left\|\frac{\partial}{\partial t}u_n\right\|\|\nabla u_n\|_\infty + |\alpha|\left\|\frac{\partial}{\partial t}u_n\right\| + |\alpha|\left\|\frac{\partial}{\partial t}v_n\right\| \le C + \left\|\frac{\partial^4 u_n}{\partial x^4}\right\| \tag{2.49}$$

So, it follows that $\int_0^t \left\|\frac{\partial^2 u_n}{\partial t^2}\right\| d\tau \le \int_0^t \left(C + \left\|\frac{\partial^4 u_n}{\partial x^4}\right\| \right)^2 d\tau \le C(T)\left(1 + \int_0^t \left\|\frac{\partial^4 u_n}{\partial x^4}\right\|^2 d\tau\right) \le C_1(T)$ (2.50)

Similarly we have $\int_0^t \left\|\frac{\partial^2 v_n}{\partial t^2}\right\| d\tau \le \int_0^t \left(C + \left\|\frac{\partial^4 v_n}{\partial x^4}\right\| \right)^2 d\tau \le C(T)\left(1 + \int_0^t \left\|\frac{\partial^4 v_n}{\partial x^4}\right\|^2 d\tau\right) \le C_2(T)$ (2.51)

Thus we have shown that the finite dimensional approximation u_n, v_n of (2.1)-(2.2) satisfies same energy bounds as u, v of a priori energy estimate.

4. Convergence of approximate solution [8]

We shall use following Lemmas and Arzela's dominated convergence theorem for the proof of convergence of approximate solution to a solution of the system of Burgers Equation (2.1)-(2.2). Arzela's dominated convergence theorem for the Riemann integral

Let $\{f_n\}$ be a sequence of Riemann-integrable functions defined on a bounded and closed interval [a, b], which converges on [a, b] to a Riemann-integrable function f .If there exist a constant $M > 0$ satisfying $\|f(x)\| \le M$ for all $x \in [a,b]$ then $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)| dx = 0$. In particular $\lim_{n\to\infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n\to\infty} f_n(x) dx = \int_a^b f(x) dx$

Lemma 4: (Tanveer, 2019a) [7]

For any fixed x, $\left\{\frac{\partial u_n(x,t)}{\partial t}\right\}_{n=1}^{\infty}$ is an equicontinuous family of functions of $t \in [0,T]$ and therefore has a subsequence that converges uniformly for $t \in [0,T]$.

Lemma 5: If $s_n \to s$ in $C^1[0,2\pi]$, then $\|P_n s_n - s\|_\infty \to 0$ as $n \to \infty$.

Lemma 6: Define $\vec{m}_n = \left(u_n, \frac{\partial u_n}{\partial x}, \frac{\partial^2 u_n}{\partial x^2}\right), \vec{l}_n = \left(v_n, \frac{\partial v_n}{\partial x}, \frac{\partial^2 v_n}{\partial x^2}\right)$. Then for $t \in [0,T]$, exist subsequences $\{\vec{m}_n\}_{j=1}^{\infty}, \{\vec{l}_n\}_{j=1}^{\infty}$ which converges in the sup norm to $\vec{m} = \left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \vec{l} = \left(v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\right)$, where u, v satisfy the integral equation

$\{(u,v)\}(x,t) = \{(u_0, v_0)\}(x,t) + \int_0^t \{(\Delta u, \Delta v) - ([u\nabla u],[v\nabla v]) + \delta(u,v)\}(x,\tau) d\tau$ Implying u, v satisfy (2.1)-(2.2)

Proof

Since $\left\|\frac{\partial^j u_n}{\partial x^j}\right\|, \left\|\frac{\partial^j v_n}{\partial x^j}\right\|$ for $j = 0,1,2$ have the bounds independent of t & n. Therefore $\{m_n\}_{n=1}^{\infty}, \{l_n\}_{n=1}^{\infty}$ forms an equicontinuous families of functions they have subsequences $\vec{m}_{n_j}$ & $\vec{l}_{n_j}$ converge in the sup norm for each $t \in [0,T]$ to $\vec{m}$ & $\vec{l}$ zero at $x = (0, 2\pi)$

We now prove that u, v satisfy the coupled system of Burgers equation. Since $u_{n_j} \to u$ in $C^2(0,2\pi)$ for each $t \in [0,T]$, it follows that

$$\left\|u_{n_j}\frac{\partial u_{n_j}}{\partial x} - u\frac{\partial u}{\partial x}\right\|_\infty \to 0, \left\|\frac{\partial}{\partial x}\left\{u_{n_j}\frac{\partial u_{n_j}}{\partial x} - u\frac{\partial u}{\partial x}\right\}\right\|_\infty \to 0 \ \& \left\|v_{n_j}\frac{\partial v_{n_j}}{\partial x} - v\frac{\partial v}{\partial x}\right\|_\infty \to 0, \left\|\frac{\partial}{\partial x}\left\{v_{n_j}\frac{\partial v_{n_j}}{\partial x} - v\frac{\partial v}{\partial x}\right\}\right\|_\infty \to 0 \quad (2.52)$$

as $j \to \infty$. Also each term in the time integral

$$\int_0^t \left\{\nabla^2 u_{n_j} - P_{n_j}\left[u_{n_j}\nabla u_{n_j} + \alpha\left(u_{n_j} - v_{n_j}\right)\right]\right\}(x,\tau) d\tau \quad (2.53)$$

is bounded independent of t in the $\sup_{x\in[0,2\pi]}$ sense.

Furthermore, using previous lemma, for any given x, subsequences $\{u_{n_j}\}_{j=1}^{\infty}$ & $\{v_{n_j}\}_{j=1}^{\infty}$ converges uniformly for $t \in [0,T]$ as $j \to \infty$

.

Therefore, from dominating convergence theorem, and using Lemma5, we have

$$\lim_{j\to\infty}\int_0^t\left\{\nabla^2 u_{n_j} - P_{n_j}\left[u_{n_j}\nabla u_{n_j} + \alpha\left(u_{n_j} - v_{n_j}\right)\right]\right\}(x,\tau)\,d\tau = \int_0^t\left\{\nabla^2 u - \left[u\nabla u + \alpha\left(u - v\right)\right]\right\}(x,\tau)\,d\tau$$
$$\lim_{j\to\infty}\int_0^t\left\{\nabla^2 v_{n_j} - P_{n_j}\left[v_{n_j}\nabla v_{n_j} + \alpha\left(v_{n_j} - u_{n_j}\right)\right]\right\}(x,\tau)\,d\tau = \int_0^t\left\{\nabla^2 v - \left[v\nabla v + \alpha\left(v - u\right)\right]\right\}(x,\tau)\,d\tau \quad (2.54)$$

Further, it is clear that $\|P_n u_0 - u_0\|_\infty \to 0, \|P_n v_0 - v_0\|_\infty \to 0$. On the other hand, $\lim_{j\to\infty} u_{n_j}(x,t) = u(x,t), \lim_{j\to\infty} v_{n_j}(x,t) = v(x,t)$ It follows from (2.34) that u, v satisfy

$$u(x,t) = u_0 + \int_0^t\left\{\nabla^2 u - \left[u\nabla u + \alpha\left(u - v\right)\right]\right\}d\tau, v(x,t) = v_0 + \int_0^t\left\{\nabla^2 v - \left[v\nabla v + \alpha\left(v - u\right)\right]\right\}d\tau$$

Which immediately implies $u(x,t), v(x,t)$ satisfy system of coupled Burgers equation with initial condition $u_0(x), v_0(x)$. Here comes the end of the proof of the Theorem.

5. Uniqueness of the solution

We use Energy method to prove uniqueness for the system of Burgers equation (1)

We work first with the first equation of the system $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + \alpha(u - v) = 0$ (2.1)*i*

Assume we had solutions $u, u+s \,\&\, v, v+s$ satisfying the equations (2.1). Then we replace u by $u+s$ and u by $v+s$ in the equation (2.1) (i) $\frac{\partial(u+s)}{\partial t} - \frac{\partial^2(u+s)}{\partial x^2} + (u+s)\frac{\partial(u+s)}{\partial x} + \alpha((u+s)-(v+s)) = 0(iii)$, subtracting equation *(i)* from *(iii)* we see that

$$\overrightarrow{m_n} = \left(u_n, \frac{\partial u_n}{\partial x}, \frac{\partial^2 u_n}{\partial x^2}\right), \overrightarrow{l_n} = \left(v_n, \frac{\partial v_n}{\partial x}, \frac{\partial^2 v_n}{\partial x^2}\right). \; S \text{ will satisfy } \frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} + \frac{\partial(us)}{\partial x} = \frac{\partial^2 s}{\partial x^2}, \quad s(x,0) = 0 \quad (2.55)$$

Multiplying (2.55) with S and integrating in $x \in R$, we obtain

$$\frac{d}{dt}\frac{1}{2}\|s\|_{L_2}^2 + \left\|\frac{\partial s}{\partial x}\right\|_{L_2}^2 = \left\langle\frac{\partial s}{\partial x}, us\right\rangle, \left\langle\frac{\partial s}{\partial x}, us\right\rangle \le \|u\|_\infty \|s\| \left\|\frac{\partial s}{\partial x}\right\| \le \frac{1}{2}\|u\|_\infty^2 \|s\|_{L_2}^2 + \frac{1}{2}\left\|\frac{\partial s}{\partial x}\right\|_{L_2}^2 \text{ This implies that } \frac{d}{dt}\frac{1}{2}\|s\|_{L_2}^2 \le \frac{1}{2}\|u\|_\infty^2 \|s\|_{L_2}^2 \quad (2.56)$$

From Gronwall's Lemma we have $\frac{1}{2}\|s\|_{L_2}^2 \le \frac{1}{2}\|s_0\|_{L_2}^2 \exp\left[\int_0^t \|u\|_{L_2}^2 d\tau\right]$ (2.57)

From (2.57) we have $s = 0$ for all t since this is the case for $t = 0$. This is true for the equation 1*(ii)* so we have unique solution in the class of solutions u, v for which $\|u\|_{L_2}^2, \|v\|_{L_2}^2$ are integrable in time. This proves the existence and uniqueness of the solution of the coupled system of Burger's equation (2.1)-(2.2).

6. Conclusion

We have proved the existence and uniqueness of solution of coupled and system of Burger's equation in one dimension. This leaves a scope of extending this study to two or more dimensional coupled system of Burger's equation and to inhomogeneous case.

7. References

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