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E-Bayesian estimation of the shape parameter of Lomax model under symmetric and asymmetric loss functions

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Abstract
This article is concerned with using the E-Bayesian method for computing estimates of the unknown parameter of Lomax distribution. These estimates are derived based on a conjugate prior for the parameter under squared error loss function and Linex loss function. A comparison between this method and the corresponding Bayes and maximum likelihood techniques is conducted using Monte Carlo simulation.

Keywords: E-Bayesian estimation, gamma prior, lomax distribution, loss functions, monte carlo simulation

1. Introduction

The Lomax distribution was first introduced by K. S. Lomax in 1954. Orginally, it was introduced for modelling of business failure data, size of cities, reliability modelling, and lifetime testing in engineering as well as in survival analysis. Hassan and Al-Ghamdi (2009) [13] used Lomax distribution for determination of optimal times of changing level of stress for simple stress plans under a cumulative exposure model. Ahmad *et al.* (2015) [4] obtained the Bayes estimators of the shape parameters of the Lomax distribution by employing the Jeffery's and extension of Jeffery's prior using Al-Bayyati's loss function, squared error loss function, and precautionary loss function.

The lomax distribution has been used in the literature in a number of ways. For example it has been extensively used for reliability modelling and life testing. Several authors have addressed inferential issues for the Lomax distribution based on complete and censored samples. The Lomax distribution has, in recent years, assumed opposition of importance in the field of life testing because of its uses to fit business failure data. It has been used in the analysis of income data, and business failure data. It may describe the lifetime of a decreasing failure rate component as a heavy tailed alternative to the exponential distribution. Abdullah and Abdullah (2010) [2] estimated the parameters of Lomax distribution based on generalized probability weighted moment. Zagan (1999) [22] deals with the properties of the Lomax distribution with three parameters. Abd-Elfatth and Mandouh (2004) [1] discussed inference for R = Pr{Y<X} when X and Y are two independent Lomax random variables. Afaq *et al.* (2015) [4] estimates the parameters of Lomax distribution using Jeffery's and extension of Jeffery's prior under different loss functions. It was first introduced for modeling business failures but latter applied to solve prediction problems arising in the fields of biological sciences, business economics, and internet traffic modelling.

Due to its broad applicability, some generalized forms of Lomax distribution were derived and studied like Exponentiated Lomax (Abdul-Moniem and Abdel-Hameed, 2012) [3], Kumaraswamy Exponentiated Lomax (Batal and Kareem, 2014) [7], Marshall-Olkin extended-Lomax (Ghitany, Al-Awadhi and Alkhalfan, 2007) [9], McDonald Lomax (Lemonte and Cordeiro, 2013) [15], Kumaraswamy - Generalized Lomax (Shams, 2013) [19], Exponential Lomax (El-Bassiouny, Abdo and Shahen, 2015) [8], Transmuted Lomax (Ashour and Eltehiwy, 2013) [6].

The Lomax distribution is a widely used distribution that has applications in the field of actuarial science, reliability modeling, life testing, economics, network analysis, and operations research. Nasiri and Hosseini (2012) [17] performs comparisons of maximum likelihood estimation (MLE) based on records and a proper prior distribution to attain a Bayes estimation (both informative and non-informative) based on records under quadratic loss and squared error loss functions. Nayak (1987) [18] suggested multivariate Lomax distribution and compute its properties and usefulness in reliability theory. The probability density function (pdf) of the Lomax distribution with two parameter is:

$$f(x:\theta,\lambda) = x,\theta,\lambda > 0 \qquad (1)$$

Where θ and λ are shape and scale parameters respectively.
This article can be organized as follows. In Section 2, we present the derivation of the maximum likelihood estimator (MLE) of the involved parameter. Bayesian estimation under squared error and LINEX loss functions is described in Section 3. Section 4 deals with the formulas of E-Bayesian estimation of the parameter under squared error and LINEX loss functions. Section 5 gives a Monte Carlo simulation study. Finally, Section 6 provides some concluding remarks.

2. Maximum likelihood estimation

Let $x_1, x_2, \ldots x_n$ is a sample taken from the Lomax distribution. Then the likelihood function for the given sample observation is

$$L=\prod_{i=1}^{n} f(x_i, \theta, \lambda)$$

$$= \left[\frac{\theta}{\lambda}\right]^n \prod_{i=1}^{n} \left[1+\frac{x_i}{\lambda}\right]^{-(\theta+1)} \qquad (2)$$

Without the additive constant, the log-likelihood function can be written as

$$\ell = n \log\theta - \theta w$$

Where

$$w=\sum_{i=1}^{n} log(1+\frac{x_i}{\lambda}) \qquad (3)$$

Thus, the corresponding likelihood equation for the parameter θ becomes

$$\frac{\partial\ell}{\partial\theta} = 0 \Longrightarrow \frac{n}{\theta} - w = 0 \qquad (4)$$

It follows, the ML estimate of θ is

$$\hat{\theta}_{ML} = \frac{n}{w} \qquad (5)$$

3. Baysian Estimation

The Bayesian philosophy is named after the Reverend Thomas Bayes (1701-1761) who made use of a subjective approach to define probability. Bayes never published his most famous attainment. His notes were edited and posthumously published by Richard Price in 1763. Often, the Bayesian statistician is free to interpret probability in whatever way best suits the problem as a frequency, as a degree of belief or simply as a function that obeys the mathematical rules of probability. Bayes made the ingenious discovery that bears his name but mysteriously abandoned it.

The Bayesian approach provides the possibility for incorporating prior information about the relevant parameters. To this end the parameter θ is considered as a random variable, having some specified distribution. Here we consider the following gamma conjugate prior density for the parameter θ.

$$\pi(\theta) = \theta^{a-1} exp\{-b\theta\}, \theta, a, b > 0 \qquad (6)$$

From 2.1 and 3.1, the posterior density of θ can be obtained as

$$\pi^*(\theta| x) = \frac{[b+w]^{n+a}}{\Gamma(n+a)} \theta^{n+a-1} exp\{-\theta(b+w)\} \qquad (7)$$

Under the squared error loss function, the Bayes estimate of θ is given by

$$\hat{\theta}_{BS}(a,b) = \frac{n+a}{b+w} \qquad (8)$$

Based on LINEX loss function, the Bayes estimate of θ can be shown to be

$$\hat{\theta}_{BL}(a,b) = \frac{-(n+a)}{k} ln\left(\frac{b+w}{a+b+w}\right) \qquad (9)$$

4. E-Bayesian Estimation

The E-Bayesian is a new criteria of estimation that was first introduced by Han (2005) [10]. This method consists of obtaining the expectation of Bayes estimates with respect to the distributions of hyper parameters. Many authors applied the E-Bayes technique such as, Han (2006) [11] estimated the reliability parameter of the exponential distribution by using the E-Bayes and hierarchical Bayes methods based on type-I censored and by considering the quadratic loss function. Yin and Liu (2010) [21] applied the E-Bayesian and hierarchical Bayesian estimation techniques to estimate the unknown reliability parameter of geometric distribution based on scaled squared loss function in complete samples. Wei *et al.* (2011) [20] obtained the minimum risk equivariant and E-Bayes estimates for the Burr-XII distribution under entopy loss function in complete samples. Jaheen and Okasha (2011) [14] estimated the parameter and reliability function of Burr-XII model via Bayes and E-Bayes techniques based on squared error and Linex loss functions on type-II censoring. Liu *et al.* (2014) [21] obtained the E-Bayes and hierarchical estimates for the Rayleigh distribution based on q symmetric entropy loss function in complete samples. Reyad *et al.* (2016) compared the Bayes, E-Bayes hierarchical Bayes and empirical Bayes estimates of shape parameter and hazard function associated to the Gompertz model under type-II censoring and by using squared error, quadratic entropy and LINEX loss functions.
The main objective of this paper is to compare the Bayes and E-Bayes methods for estimating the shape parameter corresponding to the Lomax distribution. All estimates are obtained based on symmetric and asymmetric loss function.
According to Han, the prior parameters a and b should be selected to guarantee that $\pi(\theta)$ is a decreasing function of θ. The derivative of $\pi(\theta)$ with respect to θ is

$$\frac{d\pi(\theta)}{d\theta} = \frac{b^a}{\Gamma(a)} \theta^{a-2} exp\{-b\theta\}[(a-1) - b\theta] \qquad (10)$$

Since $a > 0, b > 0$ and $\theta > 0$, it follows $0 < a < 1, b > 0$ due to

$\frac{d\pi(\theta)}{d\theta} = 0$ and therefore $\pi(\theta)$ is a decreasing function of θ. Assuming that a and b are independent with bivariate density function

$$\pi(a,b) = \pi_1(a)\,\pi_2(b) \tag{11}$$

Then, the E-Bayesian estimate of θ (expectation of the Bayesian estimate of θ) can be written as

$$\hat{\theta}_{E-B} = E(\theta|x) = \int\int \hat{\theta}_B(a,b)\,\pi(a,b)\,da\,db \tag{12}$$

Where $\hat{\theta}_B(a,b)$ is the Bayes estimate of θ under squared error and LINEX loss functions, given by (8), (9) respectively.
E-Bayesian estimation based on three different distributions of the hyper parameters a and b is obtained in this subsection, to investigate the influence of different prior distributions on the E-Bayesian estimation of θ. The following distributions of a and b may be used.

$$\pi_1(a,b) = \frac{2(c-b)}{c^2}, 0 < a < 1, 0 < b < c$$

$$\pi_2(a,b) = \frac{1}{c}, 0 < a < 1, 0 < b < c$$

$$\pi_3(a,b) = \frac{2b}{c^2}, 0 < a < 1, 0 < b < c$$

4.1 E-Bayesian estimation using squared error loss function

For $\pi_1(a,b)$, the E-Bayesian estimate of θ is

$$\hat{\theta}_{E-BSI} = \int\int \hat{\theta}_{BS}(a,b)\,\pi_1(a,b)\,da\,db$$

$$= \frac{2}{c^2}\int_0^1\int_0^C \left(\frac{n+a}{b+w}\right)(c-b)\,db\,da$$

$$=2\frac{\left(n+\frac{1}{2}\right)}{c^2}\left[(c+w)\ln\left(1+\frac{c}{w}\right)-c\right] \tag{13}$$

Similarly, the E-Bayesian estimates of θ using $\pi_2(a,b)$ and $\pi_3(a,b)$ are computed and given, respectively, by

$$\hat{\theta}_{E-BS2} = \frac{n+\frac{1}{2}}{c}\ln\left(1+\frac{c}{w}\right) \tag{14}$$

$$\hat{\theta}_{E-BS3} = \frac{2\left(m+\frac{1}{2}\right)}{c^2}\left[c-w\ln\left(1+\frac{c}{w}\right)\right] \tag{15}$$

4.2 E-Bayesian estimation using LINEX loss function

For $\pi_1(a,b)$, the E-Bayesian estimate of θ under Linex loss function

$$\hat{\theta}_{E-BLI} = \int\int \hat{\theta}_{BL}(a,b)\,\pi_1(a,b)\,da\,db$$

$$= \frac{2}{c^2}\int_0^1\int_0^C \left(-\frac{(n+a)}{k}\right)\ln\left(\frac{b+w}{b+k+w}\right)(c-b)\,db\,da$$

$$= \frac{n+\frac{1}{2}}{kc^2}\left[c^2\ln\left(1+\frac{k}{c+w}\right)-(2cw+w^2)\ln\left(1+\frac{c}{w}\right)\right.$$
$$\left.+\left((k+w)^2+2c(k+w)\right)\ln\left(1+\frac{c}{k+w}\right)-ck\right] \tag{16}$$

The E-bayesian estimates of θ based on $\pi_2(a,b)$ and $\pi_3(a,b)$ are computed and given, respectively, by

$$\hat{\theta}_{E-BL2} = \frac{n+\frac{1}{2}}{kc^2}\left[c^2\ln\left(1+\frac{k}{c+w}\right)-cw\ln\left(1+\frac{c}{w}\right)\right.$$
$$\left.+c(k+w)\ln\left(1+\frac{c}{k+w}\right)\right] \tag{17}$$

and

$$\hat{\theta}_{E-BL3} = \frac{n+\frac{1}{2}}{kc^2}\left[c^2\ln\left(1+\frac{k}{c+w}\right)+w^2\ln\left(1+\frac{c}{w}\right)\right.$$
$$\left.-(k+w)^2\ln\left(1+\frac{c}{k+w}\right)+ck\right] \tag{18}$$

5. Monte Carlo simulation study

We perform Monte Carlo simulation to compare the performances of the different estimators for different sampling schemes. Monte Carlo simulations were performed utilizing 10000 samples for each simulations. The mean squared error (MSE) is used to compare the estimators. We compute the maximum likelihood estimate and Bayes estimates respectively using (5), (8), (9). Also we compute the E-Bayesian estimates of parameter θ using (13)-(18).

Table 1: Bias of Maximum likelihood estimates of θ

n	θ	Bias
25	1.5	0.04025
	2.5	0.04333
	3.5	0.03629
30	1.5	0.04008
	2.5	0.04105
	3.5	0.03505
50	1.5	0.03548
	2.5	0.03489
	3.5	0.03478
75	1.5	0.03007
	2.5	0.03001
	3.5	0.02789

Table 2: Bias and MSE (in parentheses) for different Bayesian estimates of the parameter θ under squared error loss function and Linex loss function

		$Bias_{BL}$		$Bias_{BS}$	
25	1.5	0.02528	(0.03261)	0.02663	(0.12441)
	2.5	0.01333	(0.02061)	0.01937	(0.10251)
	3.5	0.01629	(0.01136)	0.01932	(0.09990)
30	1.5	0.01388	(0.02061)	0.01897	(0.10520)
	2.5	0.01605	(0.01196)	0.01839	(0.09629)
	3.5	0.01175	(0.00901)	0.01829	(0.07235)
50	1.5	0.01089	(0.02145)	0.01818	(0.10584)
	2.5	0.01075	(0.01851)	0.01662	(0.09067)
	3.5	0.01054	(0.00159)	0.01648	(0.07214)
75	1.5	0.01115	(0.01715)	0.01637	(0.09034)
	2.5	0.01109	(0.01654)	0.01526	(0.09014)
	3.5	0.01095	(0.00864)	0.01522	(0.07418)

Table 3: Bias and MSE (in parentheses) for different E-Bayesian estimates of the parameter θ under squared error loss function

n	θ	$Bias_{E-BLI}$		$Bias_{E-BL2}$		$\hat{\theta}_{E-BL3}$	
	1.5	0.01699	(0.03766)	0.01569	(0.07569)	0.01577	(0.01942)
25	2.5	0.01698	(0.02621)	0.01517	(0.05853)	0.0125	(0.01632)
	3.5	0.01682	(0.02185)	0.01501	(0.04755)	0.01469	(0.00634)
	1.5	0.01672	(0.02718)	0.01431	(0.05839)	0.01439	(0.01327)
30	2.5	0.01650	(0.02482)	0.01409	(0.03071)	0.01427	(0.01239)
	3.5	0.01600	(0.01119)	0.01401	(0.02213)	0.01417	(0.00759)
	1.5	0.01593	(0.03087)	0.01322	(0.03657)	0.01401	(0.01263)
50	2.5	0.01580	(0.02106)	0.01399	(0.02954)	0.01387	(0.01218)
	3.5	0.01579	(0.01147)	0.01338	(0.01765)	0.01346	(0.00628)
	1.5	0.01460	(0.02157)	0.01325	(0.02631)	0.01337	(0.00942)
75	2.5	0.01364	(0.02009)	0.01323	(0.05161)	0.01332	(0.01899)
	3.5	0.01030	(0.01118)	0.01312	(0.02604)	0.01222	(0.01742)

Table 4: Bias and MSE (in parentheses) for different E-Bayesian estimates of the parameter θ under Linex loss function

n	θ	$Bias_{E-BLI}$		$Bias_{E-BL2}$		$Bias_{E-BL3}$	
	1.5	0.01585	(0.00259)	0.01544	(0.00389)	0.01436	(0.00052)
25	2.5	0.01433	(0.00064)	0.01392	(0.01597)	0.01284	(0.02001)
	3.5	0.01377	(0.0036)	0.01136	(0.01041)	0.01428	(0.01049)
	1.5	0.01147	(0.00059)	0.01106	(0.00589)	0.01198	(0.02003)
30	2.5	0.01125	(0.01057)	0.01084	(0.01027)	0.01106	(0.01098)
	3.5	0.01115	(0.00006)	0.01034	(0.00088)	0.01126	(0.00061)
	1.5	0.01116	(0.00404)	0.01098	(0.01162)	0.01090	(0.00185)
50	2.5	0.01111	(0.00156)	0.01074	(0.01833)	0.01066	(0.00013)
	3.5	0.01089	(0.00201)	0.01063	(0.00639)	0.01095	(0.00805)
	1.5	0.01086	(0.00106)	0.01082	(0.00452)	0.01076	(0.01303)
75	2.5	0.01071	(0.00597)	0.01069	(0.00896)	0.01091	(0.01467)
	3.5	0.01068	(0.00859)	0.01064	(0.00324)	0.01001	(0.00292)

6. Concluding remarks

In this article, we considered the maximum likelihood, Bayesian and E- bayesian estimates for the shape parameter of Lomax distribution. Based on results shown in Table 1, Table 2, Table 3 and Table 4 we observe the following:

1. Bayesian and E-bayesian estimators perform much better than the maximum likelihood estimator in terms of MSEs.
2. E-Bayesian estimators perform better than the Bayesian estimator.
3. The bias reduces as the sample size increases.
4. The estimate under Linex loss function has lesser bias than the squared error loss.

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