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KB method for obtaining an approximate solution of slowly varying amplitude and phase of nonlinear differential systems with varying coefficients

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Abstract

To determine an approximate solutions of damped nonlinear ordinary differential system with varying coefficients of oscillatory procedure is envisaged based on the Krylov-Bogoliubov (KB) method. Our aim is to this paper of Krylov and Bogoliubov (KB) for determining an approximate solution of a nonlinear differential system with varying coefficients. Finally, results are discussed, especially to enrich the physical prospects, and shown graphically by utilizing Mathematica and Matlab software. However, in some cases it's feasible to alternate nonlinear differential equations with an associated linear equation closely enough to give helpful results. Some expository examples are given to exhibit the suitability and proficiency considered method.

Keywords: Non-linear equations, KB method, Perturbation methods, slowly varying coefficients

Introduction

The mathematical formulation of such problems give rise to differential equation and also within the mathematical formulations, many physical issues usually end in differential equations that are nonlinear. The process of mathematical formulation, certain simplifying assumptions generally has to be made in order that the resulting differential equations are tractable. Arya and Bojadziev (1980) [1] studied a system of second order nonlinear hyperbolic differential equation with slowly varying coefficients. The method has been extended to damped oscillatory and purely non oscillatory systems with slowly varying coefficients by Bojadziev and Edwards (1981) [2]. This method was developed originally by Krylov and Bogoliubov and it first appeared in published form in 1937. The method has been extended and justified mathematically by Bogoliubov and Mitropolsky. These authors call their method asymptotic in the sense that $\epsilon \rightarrow 0$. The advantage of the method is that it not only enables us to determine the steady-state periodic motions but also allows us to determine the transient process corresponding to perturbations of these oscillations. Bojadziev, G.N (1972) On Asymptotic solutions of nonlinear differential equations with time lag, delay and functional differential equations and their applications and Bojadziev, G.N., The Krylov-Bogoliubov-Mitropolskii mehod (1978) applied to models of population dynamics, Bulletin of Mathematical Biology. In this chapter we consider a method of treating weakly nonlinear differential equations of the form

$$\frac{d^2y}{dt^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right) \quad 0 < \epsilon \ll 1 \tag{1.1}$$

$$\frac{d^2y}{dt^2} - 2p \frac{dy}{dt} + q^2 y = \epsilon F\left(y, \frac{dy}{dt}\right) \quad 0 < \epsilon \ll 1 \tag{1.2}$$

Where
 p and q are real constants, and $F(y, dy/dt)$ is a known nonlinear function.

Method and Materials

Let us consider the method of determining an approximate solution of a nonlinear differential equation having the form

$$\frac{d^2y}{dt^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right) \quad 0 < \epsilon \ll 1 \quad \text{If } \epsilon = 0, \quad (2.1)$$

Then equation (2.1) reduces to the linear equation

$$\frac{d^2y}{dt^2} + y = 0 \quad (2.2)$$

The solution of equation (2.2) may be written

$$y = a \cos(t + \varphi) \quad (2.3)$$

Where a and φ are constants. Note that the derivative of the solution given by equation (2.3) is

$$\frac{dy}{dt} = -a \sin(t + \varphi) \quad (2.4)$$

If $\epsilon \neq 0$, but is sufficiently small. One can assume that the nonlinear equation (2.1) also has a solution of the form of equation (2.3) with derivative of the form of equation (2.4) provided that a and φ are functions of t rather than being constants. That is, we assume a solution of equation (2.3) of the form

$$y = a(t) \cos[t + \varphi(t)] \quad (2.5)$$

Where a and φ are functions of t to be determined such that the derivative of the solution, equation (2.5) is of the form

$$\frac{dy}{dt} = -a(t) \sin[t + \varphi(t)] \quad (2.6)$$

Differentiating this assumed solution, we obtain

$$\frac{dy}{dt} = \frac{da}{dt} \cos[t + \varphi(t)] - a \sin[t + \varphi(t)] - a \frac{d\varphi}{dt} \sin[t + \varphi(t)] \quad (2.7)$$

In order for $\frac{dy}{dt}$ to have the form given by equation (2.6) we must require

$$\left[\frac{da}{dt} \cos[t + \varphi(t)] \right] - a \frac{d\varphi}{dt} \sin[t + \varphi(t)] = 0 \quad (2.8)$$

Differentiating the assumed derivative equation (2.8), we obtain

$$\frac{d^2y}{dt^2} = -\frac{da}{dt} \sin(t + \varphi) - a \cos(t + \varphi) - a \frac{d\varphi}{dt} \cos(t + \varphi) \quad (2.9)$$

Substituting the assumed solution, equation (2.5) its assumed derivative, equation (2.6) and the second derivative given by equation (2.9) into the differential equation (2.1), we obtain

$$\begin{aligned} & -\frac{da}{dt} \sin(t + \varphi) - a \cos(t + \varphi) - a \frac{d\varphi}{dt} \cos(t + \varphi) + a \cos(t + \varphi) \\ & = \epsilon F[a \cos(t + \varphi), -a \sin(t + \varphi)] \end{aligned} \quad (2.10)$$

Or

$$\frac{da}{dt} \sin(t + \varphi) + a \frac{d\varphi}{dt} \cos(t + \varphi) = -\epsilon F[a \cos(t + \varphi) - a \sin(t + \varphi)]$$

If we let t denotes $t + \varphi(t)$, then equations (2.10) and (2.13) may be written as

$$\frac{da}{dt} \cos \Psi(t) - a \frac{d\varphi}{dt} \sin \Psi(t) = 0 \quad (2.11)$$

$$\frac{da}{dt} \sin \Psi(t) + a \frac{d\varphi}{dt} \cos \Psi(t) = -\epsilon F(a \cos \Psi, -a \sin \Psi) \quad (2.12)$$

Solving the system given by equation (2.12) for $\frac{da}{dt}$ and $\frac{d\varphi}{dt}$, we obtain the following equations:

$$\begin{aligned}\frac{da}{dt} &= -\epsilon F[a(t) \cos \Psi(t) - a(t), \sin \Psi(t)] \sin \Psi(t) \\ \frac{d\varphi}{dt} &= -\left[\frac{\epsilon}{a(t)}\right] F[a(t) \cos \Psi(t), -a(t) \sin \Psi(t)] \cos \Psi(t)\end{aligned}\quad (2.13)$$

These are the exact equations for the functions a and φ when the solution is of the form given by equation (2.7) with the derivative having the form of equation (2.8). Note that these equations are nonlinear and quite complicated. We now apply the first approximation of Krylov and Bogoliubov.

Expanding

$$F(a \cos \Psi, -a \sin \Psi) \sin \Psi \text{ and } F(a \cos \Psi, -a \sin \Psi) \cos \Psi$$

in Fourier series, we obtain $F \sin \Psi = K_0(a) + \sum_{n=1}^{\infty} [K_n(a) \cos n\Psi + L_n(a) \sin n\Psi]$ $F \cos \Psi = P_0(a) + \sum_{n=1}^{\infty} [P_n(a) \cos n\Psi + Q_n(a) \sin n\Psi]$

Where

$$\begin{aligned}K_0(a) &= \frac{1}{2\pi} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \sin \Psi \, d\Psi \\ K_n(a) &= \frac{1}{\pi} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \sin \Psi \cos n\Psi \, d\Psi \\ P_0(a) &= \frac{1}{2\pi} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \cos \Psi \, d\Psi \\ P_n(a) &= \frac{1}{\pi} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \cos \Psi \cos n\Psi \, d\Psi \\ L_n(a) &= \frac{1}{\pi} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \sin \Psi \sin n\Psi \, d\Psi \\ Q_n(a) &= \frac{1}{\pi} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \cos \Psi \sin n\Psi \, d\Psi\end{aligned}\quad (2.14)$$

Thus, equations (2.13) may be written

$$\begin{aligned}\frac{da}{dt} &= -\epsilon K_0(a) - \epsilon \sum_{n=1}^{\infty} [K_n(a) \cos n\Psi + L_n(a) \sin n\Psi] \\ \frac{d\varphi}{dt} &= -\frac{\epsilon}{a} P_0(a) - \frac{\epsilon}{a} \sum_{n=1}^{\infty} [P_n(a) \cos n\Psi + Q_n(a) \sin n\Psi]\end{aligned}\quad (2.15)$$

The first approximation of Krylov and Bogoliubov consists of neglecting all the terms on the right-hand side of equation (2.18) except for the first: that is.

$$\frac{da}{dt} = -\epsilon K_0(a) = -\left(\frac{\epsilon}{2a}\right) \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \sin \Psi \, d\Psi \quad (2.18)$$

$$\frac{d\varphi}{dt} = -\left(\frac{\epsilon}{a}\right) P_0(a) = \frac{\epsilon}{2\pi a} \int_0^{2\pi} F(a \cos \Psi, -a \sin \Psi) \cos \Psi \, d\Psi \quad (2.16)$$

A justification for this procedure is as follows. To solve equation (2.15) we note that the right-hand sides are periodic with respect to the variable Ψ , with a period equal to 2π .

$$\frac{da}{dt} = Q(\epsilon) \text{ and } \frac{d\varphi}{dt} = Q(\epsilon) \quad (2.17)$$

Thus, a and φ are slowly varying functions of time because ϵ is small; Hence, they change very little during the time $T = 2\pi$, the period of the terms on the right-hand side of equation (2.15). Averaging the right-hand side of equation (2.15) over the interval 2π in Ψ , during which a and φ can be taken to be constants, we obtain equation (2.19). We may also apply this procedure to equation (2.18) and obtain similar results. Integrating the first equation in equation (2.17) between the limits t and $t+T$, where $a(t)$ and $\varphi(t)$ are considered as remaining approximately constant in this interval, we obtain

$$a(t+T) - a(t) = -\epsilon K_0(a) \quad (2.18)$$

Where we have used the result

$$\int_t^{t+T} \cos m\Psi dt = \int_t^{t+T} \sin m\Psi dt = 0 \quad (2.19)$$

With $\Psi = t + T$. Under these conditions, we have

$$\frac{a(t+T)-a(t)}{T} = \frac{da}{dt} \quad (2.20)$$

and consequently obtain

$$\frac{da}{dt} = -\epsilon K_0(a) \quad (2.21)$$

The same procedure may be applied to the second equation of equations (2.18) to obtain

$$\frac{\varphi(t+T)-\varphi(t)}{T} = \frac{d\varphi}{dt} = -\left(\frac{\epsilon}{a}\right) P_0(a) \quad (2.22)$$

It should be pointed out that we have not given a rigorous justification of this procedure. This justification comes naturally with the method of Krylov Bogoliubov and Mitropolsky, which we discuss in section (2.4). Assuming that the procedure is justifiable for the problem under consideration, a first approximation to the solution of equation (2.3) is thus given by $y(t) = a(t) \cos[t + \varphi(t)]$, where $a(t)$ and $\varphi(t)$ are determined by equation (2.17)

Example 1:

Find the first approximation of the differential equation

$$\frac{d^2y}{dt^2} + y + \epsilon y^2 = 0$$

using Krylov Bogoliubov techniques Given that,

$$\frac{d^2y}{dt^2} + y + \epsilon y^2 = 0 \quad (2.23)$$

where $F = -y^2$

The non-linear function depends only on y . The equation for $a(t)$ becomes,

$$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right) \int_0^{2\pi} -a^2 \cos^2 \Psi d\Psi \quad (2.24)$$

Let $u = a \cos \Psi$

$du = -a \sin \Psi d\Psi$

When

$\Psi = 0$, then $u = a$

When

$\Psi = 2\pi$, then $u = a$

$$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right) \int_0^{2\pi} -u^2 \frac{du}{a}$$

$$\frac{da}{dt} = 0 \quad (2.25)$$

$a(t) = A = \text{constant}$ [integrating]

The equation for $\varphi(t)$ is

$$\frac{d\varphi}{dt} = \left(-\frac{\epsilon}{2\pi A}\right) \int_0^{2\pi} -A^2 \cos^2 \Psi \cos \Psi d\Psi$$

$$= \left(\frac{\epsilon}{2\pi A}\right) \int_0^{2\pi} \frac{4}{4} A^2 \cos^3 \Psi d\Psi$$

$$\begin{aligned}
 &= \frac{\epsilon A}{2\pi} \cdot \frac{1}{4} \int_0^{2\pi} (3 \cos \Psi + \cos 3\Psi) d\Psi \\
 &= \frac{\epsilon A}{8\pi} \cdot 0 = 0
 \end{aligned}
 \tag{2.26}$$

$\varphi(t) = \varphi_0 = \text{constant}$. Hence the solution of equation (2.3) using the first approximation of KB Method is

$$y = A \cos(t + \varphi_0) \tag{2.27}$$

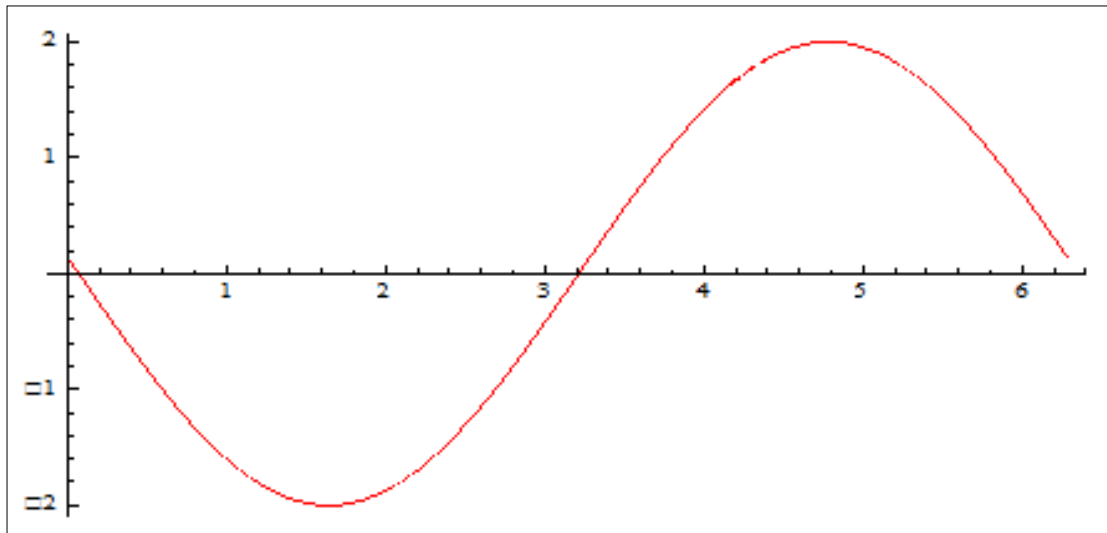


Fig 1: Damped nonlinear oscillation with initial conditions $[y(0)=1, y'(0) = 0, A = 2, \varphi = 1.5]$

Another Example 2

Find the first approximation of the differential equation $\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0$ using Krylov Bogoliubov techniques

Given that,

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0 \tag{2.28}$$

where $F = -y^3$

The non-linear function depends only on y . The equation for $a(t)$ becomes,

$$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right) \int_0^{2\pi} -a^3 \cos^3 \Psi \sin \Psi d\Psi \tag{2.29}$$

Let $u = a \cos \Psi$

$$du = -a \sin \Psi d\Psi$$

when $\Psi = 0$, then $u = a$

when $\Psi = 2\pi$, then $u = a$

then, $\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right) \int_a^a u^3 \frac{du}{a}$

$$\frac{da}{dt} = 0 \tag{2.30}$$

$a(t) = A = \text{constant}$ [Integrating]

Now, equation for $\varphi(t)$ is

$$\begin{aligned}
 \frac{d\varphi}{dt} &= \left(-\frac{\epsilon}{2\pi A}\right) \int_0^{2\pi} -A^3 \cos^3 \Psi \cos \Psi d\Psi \\
 &= \left(\frac{\epsilon A^2}{2\pi}\right) \int_0^{2\pi} \frac{1}{4} (4 \cos^4 \Psi) d\Psi
 \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\epsilon A^2}{8\pi}\right) \int_0^{2\pi} (2\cos^2\Psi)^2 d\Psi \\
&= \left(\frac{\epsilon A^2}{8\pi}\right) \int_0^{2\pi} (1 + \cos 2\Psi)^2 d\Psi \\
&= \left(\frac{\epsilon A^2}{8\pi}\right) \int_0^{2\pi} (1 + 2\cos 2\Psi + \cos^2 2\Psi) d\Psi \\
&= \left(\frac{\epsilon A^2}{8\pi}\right) \int_0^{2\pi} \left[1 + 2\cos 2\Psi + \frac{1}{2}(1 + \cos 4\Psi)\right] d\Psi \\
&= \left(\frac{\epsilon A^2}{8\pi}\right) \int_0^{2\pi} \left[\frac{3}{2} + 2\cos 2\Psi + \frac{1}{2}(\cos 4\Psi)\right] d\Psi \\
&= \left(\frac{\epsilon A^2}{8\pi}\right) \frac{1}{2} \int_0^{2\pi} [3 + 4\cos 2\Psi + \cos 4\Psi] d\Psi \\
&= \left(\frac{\epsilon A^2}{16\pi}\right) 6\pi \\
\frac{d\varphi}{dt} &= \left(\frac{3\epsilon A^2}{8}\right) \tag{2.31}
\end{aligned}$$

$$\varphi(t) = \left(\frac{3\epsilon A^2}{8}\right)t + \varphi_0$$

Where $\varphi_0 = \text{constant}$ (2.32)

Hence the solution of equation the first approximation of Krylov and Bogoliubov is

$$y = A\cos\left[t + \left(\frac{3\epsilon A^2}{8}\right)t + \varphi_0\right] = A\cos\left[\left(1 + \frac{3\epsilon A^2}{8}\right)t + \varphi_0\right] \tag{2.33}$$

Here, the amplitudes is constant and the frequency (ω) is $\left(1 + \frac{3\epsilon A^2}{8}\right)$ i.e. $\omega = 1 + \frac{3\epsilon A^2}{8}$

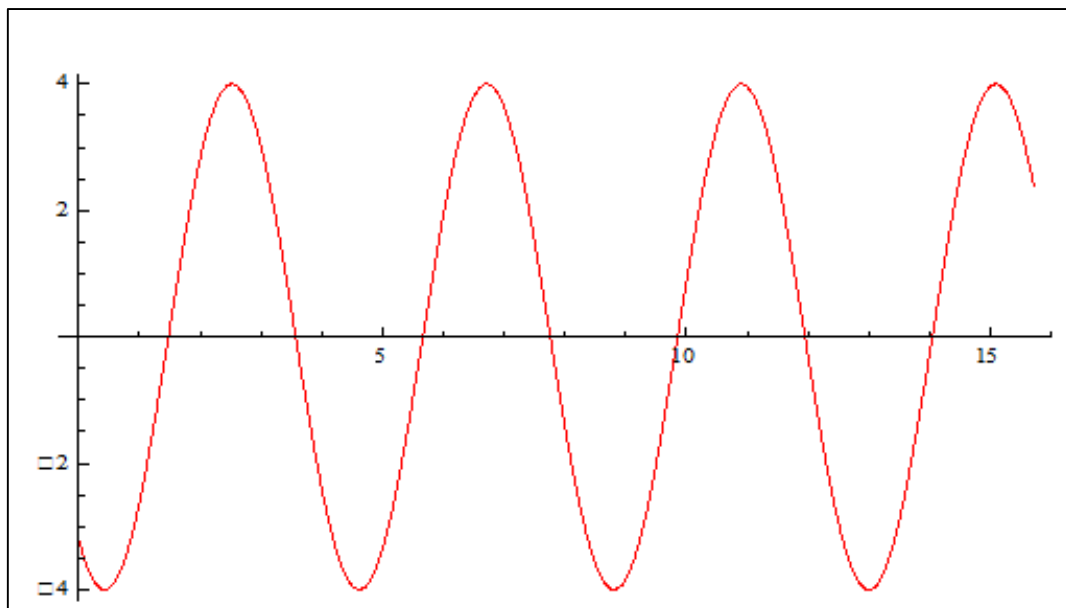


Fig 2: Damped nonlinear oscillation with initial conditions $[y(0)=1, y'(0) = 0, A = 2, \varphi = 1.5]$

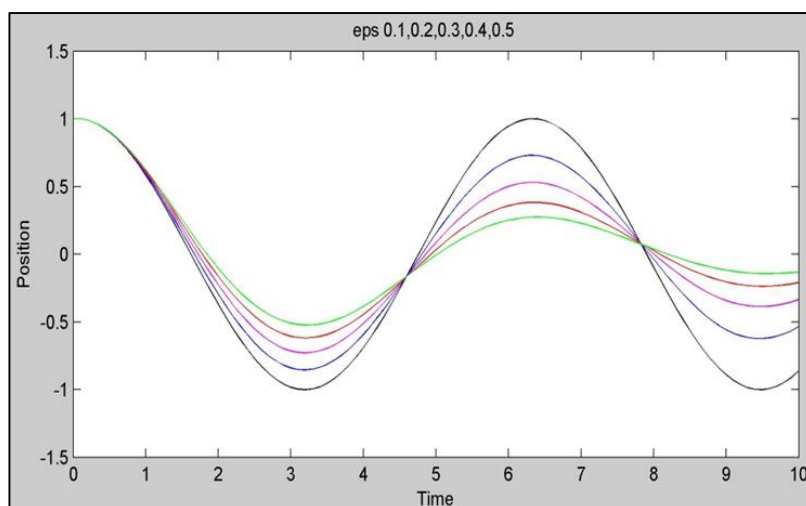
Graphically Representation of Examples 1 & 2

Fig 3: Damped linear oscillation with corresponding numerical solution [Euler] are plotted with initial condition [$y(0) = 1, y'(0) = 0, \epsilon = 0.1, \epsilon = 0.2, \epsilon = 0.3, \epsilon = 0.4, \epsilon = 0.5$]

Results and discussions

Krylov-Bogoliubov (KB) method is applied to certain damped nonlinear systems with slowly varying amplitude and phase. It represents the relationship between a continuously varying quantity and its rate of change. The equation (1.1) is the differential equations of the first approximation in the form in which they are generally used in applications. This method, though it is restricted to differential equations of type (1.1) has been used extensively in plasma physics, theory of oscillations and control theory. Finally KB Method is used to determine the stationary amplitudes and their stabilities. Thus a and φ are slowly varying function of time because ϵ is small; hence they change is very little during the time $T = 2\pi$. Any non-stationary oscillation approaches a stationary one with passage of time. The method is illustrated by the examples and Damped nonlinear oscillation with initial conditions the first approximate solutions are plotted along of that equation in Figs 1, 2. Also we see that Figure 3 Damped linear oscillation with corresponding numerical solution [Euler] graphically comparing by examples 1 & 2 amplitude and phase corresponding by KB Method.

Conclusion

In this article we find an approximate solution of a nonlinear differential system with slowly varying coefficients under the action of several damping forces. The solution is simpler than classical KB method. Here it is found that if the damping force is an important, the resolution is stable.

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