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Generic sub manifold of quaternion Kaehler manifold

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Abstract
The aim of the present paper is the analysis of a generic sub-manifold in the quaternion Kaehler manifold. Section 1 is the historical background of Kaehler and quaternion manifold. Next, we add some simple formulas and concepts that we use in research work. Again section 3 is about the integrability of generic sub-manifold in quaternion manifold. Further, in the end, area, some fundamental results are obtained concerning the parallel canonical structure in quaternion Kaehler manifold.

Keywords: Quaternion Kaehler manifold, generic submanifolds, hermitian structure, cr-submanifold, riemannian manifold

Introduction

It is well known that any quaternion-Kaehler manifold is a connection-oriented Riemannian 4n-manifold, $n \geq 2$ with holonomy group in $S_p(n)S_p(1)$, $n \geq 2$, A group that appears on the database, classification theorem given by Berger's [4]. For possible holonomy of non-symmetric, non-reducible multiple groups.

The geometry of Quaternion-Kaehler comes naturally from the concept of potential holonomies. Einstein also is the quaternion-Kaehler manifold a quaternion Kaehler manifold has a positive scalar curvature, it is considered positive. The sign of the scalar curvature gives this class of manifolds a coarse classification; when $s = 0$. The scalar curvature case is stringently distinct from zero, and thoroughly characterizes the geometry of quaternion-Kaehler. A submanifold of $\overline{M}$ called quaternion CR-submanifold [8] if there exits two orthogonal complementary distribution D and $D^{\perp}$ on M such, that D is invariant under $J_i D \subset D, i = 1,2,3$ and $D^{\perp}$ is real. The quaternion CR-submanifold includes quaternion and a completely real submanifold as subcases. A submanifold M of a quaternion Kaehler manifold is called Qr-submanifolds [2] if there exists a vector subbundle of the normal bundle $TM^{\perp}$ of M such that we have $J_i(v_p) = v_p$ and $J_i(v_p^{\perp}) \subset T_pM$ for each $p \in M$, where $v_p^{\perp}$ is the complementary orthogonal subbundle to v in $TM^{\perp}$ .it is known that every real hypersurface of a quaternion Kaehler manifold is a QR-submanifold [9]. In comparison, A. Bejancu [1] defines a CR-submanifold in Kaehler manifold, and studies it. Several articles have appeared on this subject since then [3, 5, 6, 7]. In addition, the quaternion CR-submanifold of a quaternion Kaehler manifold was described by Barros, Chen and Urbano as an analogy with the CR-submanifold of a Kaehler manifold [1]. The geometry of a submanifold of a quaternion Kaehler manifold is mainly based on the action of the local basis $\{J_1, J_2, J_3\}$ on each tangent space to M . We know that if the manifold is non-Kaehler than it is non-integrable. However, the concept of a generic submanifold of quaternion Kaehler manifolds has not been studied as yet, as so far we know. Therefore in this paper, we study the Generic submanifold of quaternion Kaehler manifold.

Preliminaries

Let $\overline{M}$ be a 4m-dimensional Riemannian manifold with a 3-dimensional vector bundle V consisting of the tensor of type (1,1) with a local basis of almost Hermitian structure $\{J_1, J_2, J_3\}$ (i.e. $g(J_iX, J_iY) = g(X,Y)$, $i = 1,2,3$) such that

(2.1) (a) $J_1^2 = J_2^2 = J_3^2 = -I$

$J_1 o J_2 = -J_2 o J_1 = J_3$,

$J_2 o J_3 = -J_3 o J_2 = J_1$

$J_3 o J_1 = -J_1 o J_3 = J_2$

I Denoting the identity tensor of type (1,1) in $\overline{M}$.

(b) If ϕ is any local cross-section (Local or global) of V then $\overline{\nabla}_X \phi$ is also a cross-section of V.

Where $\overline{\nabla}$ denotes the Riemannian connection on $\overline{M}$. And X being any arbitrary vector field on $\overline{M}$.

We see that condition (b) is the equivalent of the following:

(2.2) (b) $\overline{\nabla}_X J_1 = r(X)J_2 - q(X)J_3$

$\overline{\nabla}_X J_2 = -r(X)J_1 - p(X)J_3$

$\overline{\nabla}_X J_3 = q(X)J_1 - p(X)J_2$

And

(2.2) (a) $\tilde{\nabla}_C J_i = \sum_{k=1}^{3} Q_{ik}(C) J_i, \; i = 1,2,3$

For all vector fields C tangent to $\overline{M}$, Where $\tilde{\nabla}$ is the Levi-Civita connection and Q_{ik} are certain 1-forms locally defined on $\overline{M}$ such that $Q_{ik} + Q_{ki} = 0$

Agreement 2.1: let us define a tensor field G which satisfies:

(2.3) $G(X, \Upsilon) = -G(\Upsilon, X)$

(2.4) $G(X, J_i\Upsilon) = -J_i G(X, \Upsilon)$, $i = 1,2,3$

(2.5) $g(G(X, \Upsilon), Z) = -g(G(X, Z), \Upsilon)$

(2.6) $g(G(X, \Upsilon), Z) = g(G(\Upsilon, Z), X) = g(G(Z, X), \Upsilon)$

Let M be a sub-manifold of a quaternion space generated by $\overline{M}(c)$ and g is the metric for both $\overline{M}(c)$ an M. TM and $T^{\perp}M$ are the tangent bundle and the normal bundle of M respectively. If ∇ denote the Riemannian connection induced on M and $\nabla^{\perp}$ indicate the relationship in the normal bundle $T^{\perp}M$. The Gauss and Weingarten formulas are

(2.7) $\overline{\nabla}_X \Upsilon = \nabla_X \Upsilon + h(X, \Upsilon)$ and

(2.8) $\overline{\nabla}_X \xi = -A_\xi X + \nabla^{\perp} \xi$

For any X, Y tangent to M and any vector field ξ normal to M.

The second fundamental form h and the shape operator A_ξ are connected by

(2.9) $g(h(X, \Upsilon), \xi) = g(A_\xi X, \Upsilon)$

For the second fundamental form h, we define the covariant differentiation $\overline{\nabla}$ concerning the connection in $TM \oplus T^{\perp}M$ by

(2.10) $(\overline{\nabla}_X h)(\Upsilon, Z) = \nabla^{\perp} h(\Upsilon, Z) - h(\nabla_X \Upsilon, Z) - h(\Upsilon, \nabla_X Z)$

For any vector field X, Y, and Z tangent to M.
The equations of Gauss and Codazzi and Ricci are then respectively given by

(2.11) $R(X, \Upsilon, Z, W) = \overline{R}(X, \Upsilon, Z, W) + g(h(X, W), h(\Upsilon, Z)) - g(h(X, Z), h(\Upsilon, W))$

(2.12) $(R(X, \Upsilon), Z)^{\perp} = (\overline{\nabla}_X h)(\Upsilon, Z) - (\overline{\nabla}_\Upsilon h)(X, Z)$

(2.13) $\overline{R}(X, \Upsilon; \xi, \eta) = R^{\perp}(X, \Upsilon, \xi, \eta) - g\left(\left[A_\xi, A_\eta\right]X, \Upsilon\right)$

For any vector fields $X, \Upsilon, Z,$ and W tangent to M, ξ and η are normal to M. let us we put

(2.14) $J_i X = E_i X + F_i X$

Where the tangential and regular part of $J_i X$ is $E_i X$ and $F_i X$.

Similarly for any $\xi \in T^{\perp}M$, we have

(2.15) $J_i \xi = B_i \xi + C_i \xi$

Where $B_i \xi$ denotes the tangential component and $C_i \xi$ represents the standard component of $J_i \xi$.
For a sub-manifold M of $\overline{M}$, let us define The holomorphic tangent space of M at P as,

(2.16) $\chi_P = T_P M \cap J_i T_P M$, $i = 1, 2, 3$

Where χ_P is the maximal complex subspace of $T_P \overline{M}$.

Definition 2.1: if dim of χ_P is constant along with submanifold M in a quaternion Kaehler manifold, then M is called a generic submanifold. Where χ_P defines as a holomorphic distribution.

Definition 2.2: A generic sub-manifold M in a quaternion Kaehler manifold is a real (or Complex) sub-manifold if $J_i TM \subseteq T^{\perp}M$, $i = 1, 2, 3$.

The generic sub-manifold M in a quaternion manifold $\overline{M}$, Complementary Orthogonal distribution χ_P, to be the purely real distribution, if

(2.17) $\chi_P \perp \chi_P$, $T_i \chi_P^{\perp} \subseteq \chi_P^{\perp}$, $\chi_P^{\perp} \cap J_i \chi_P^{\perp} = \{0\}$, $i = 1, 2, 3$

From (1.12), Normal-bundle-Valued F_i indicated an isomorphism from $\chi_P^{\perp}$ onto $F_i \chi_P^{\perp}$. Let vector space v_p of normal holomorphic vectors to M at P; that is

(2.18) $v_P = T_P^{\perp} M \cap J_i T_P^{\perp} M$

Then v_P is a diff. Vector sub-bundle of $T^{\perp}M$. Infers

(2.19) $T^{\perp}M = F_i\chi^{\perp} \oplus v$, $B_i(T^{\perp}M) = \chi^{\perp}$, $g(F_i\chi^{\perp}, v) = 0$, $i = 1, 2, 3$

3. Integrability of Generic Submanifolds in Quarternion Kaehler Manifold:

Lemma 3. 1: if M be any generic sub-manifold for a quaternion Kaehler Manifold $\overline{M}$, satisfies

(3.1) $g(h(X, J_i\Upsilon) - h(J_iX, \Upsilon), \eta) = g(2G(X, \Upsilon)^{\perp}, \eta)$

For any vector $X, \Upsilon \in \chi$ and $\eta \in v$.

Theorem 3.1: any generic submanifold M concerning quaternion Kaehler manifold $\overline{M}$. Then the holomorphic distribution χ is integrable if and only if

(3.2) $g(h(X, J_i\Upsilon) - h(J_iX, \Upsilon), F_iZ) = g(2G(X, \Upsilon)^{\perp}, F_iZ)$

For vector field $X, \Upsilon \in \chi$ and $Z \in \chi^{\perp}$.

Proof: Since $\overline{M}$ is a quaternion Kaehler manifold, using (2.3) and (2.7) we have

(3.3) $h(X, J_i\Upsilon) - h(J_iX, \Upsilon) = \overline{\nabla}_X J_i\Upsilon - \nabla_X J_i\Upsilon + \nabla_\Upsilon J_iX$

$= G(X, \Upsilon) + J_i\overline{\nabla}_X\Upsilon - G(\Upsilon, X) - J_i\overline{\nabla}_\Upsilon X + \nabla_\Upsilon J_iX - \nabla_X J_i\Upsilon$

$= 2G(X, \Upsilon) + J_i[X, \Upsilon] + \nabla_\Upsilon J_iX - \nabla_X J_i\Upsilon$

So we get

(3.4) $h(X, J_i\Upsilon) - h(J_iX, \Upsilon) - 2G(X, \Upsilon) = J_i[X, \Upsilon] + \nabla_\Upsilon J_iX - \nabla_X J_i\Upsilon$

for vector field X, Y in χ .
the holomorphic distribution χ is integrable, then RHS of (3.4) lies in TM ; thus we obtain

(3.5) $h(X, J_i\Upsilon) - h(J_iX, \Upsilon) = 2G(X, \Upsilon)^{\perp}$

So we have the theorem.

Proposition 3.1: in a quaternion Kaehler manifold $\overline{M}$, M be a generic submanifold of $\overline{M}$. If χ is integrable and its leaves are geodesic in M , then

(3.6) $g((G + J_ih)(\chi, \chi), \chi^{\perp}) = 0$

Proof: its known that χ is integrable, and M is geodesic in its branches. We have $\nabla_X\Upsilon \in \chi$, for any $X, \Upsilon \in \chi$. So we have

(3.7) $g(\nabla_X Z, \Upsilon) = g(\nabla_X \Upsilon, Z) = 0$ so we can get $\nabla_X Z \in \chi^{\perp}$ for any vector $X \in \chi$ and $Z \in \chi^{\perp}$.

From (2.3), (2.7), (2.8) and (2.14) we get

(3.7) $$g(\nabla_X Z, J_i \Upsilon) = -gh(J_i \overline{\nabla}_X Z, \Upsilon) = g(G(X, Z) - \overline{\nabla}_X J_i Z, \Upsilon)$$

$$= g(G(X, Z), \Upsilon) - g(\overline{\nabla}_X EZ, \Upsilon) - g(\overline{\nabla}_X FZ, \Upsilon)$$

$$= g(G(X, Z), \Upsilon) - g(\nabla_X EZ, \Upsilon) + g(A_{FZ} X, \Upsilon)$$

Which implies that,

(3.8) $$-g(G(X, \Upsilon), Z) + g(h(X, \Upsilon), J_i Z) = -g(G(X, \Upsilon), Z) = 0$$

The proposition is proved.

Theorem 3.2: any generic sub-manifold M of a quaternion Kaehler manifold $\overline{M}$. If $\chi^{\perp}$ is integrable and its leaves are geodesic in M, then

(3.9) $$g\left((G - J_i h)(\chi, \chi^{\perp}), \chi^{\perp}\right) = 0$$

Proof: it follows from (2.4), (2.7), (2.8) and (2.14) that

(3.10) $$g(_Z X, W) = g(J_i \overline{\nabla}_Z X, J_i W) = g(-G(Z, X) + \overline{\nabla}_Z J_i X, EW + FW)$$

$$= g(G(X, Z), J_i W) + g(\overline{\nabla}_Z J_i X, EW) + g(\overline{\nabla}_Z J_i X, FW)$$

$$= g(G(X, Z), J_i W) + g(h(J_i X, Z), FW)$$

$$= -g(J_i G(X, Z) + J_i h(J_i X, Z), W) = g(G(J_i X, Z) - J_i h(J_i X, Z), W)$$

That is

(3.11) $g(G(J_i X, Z) - J_i h(J_i X, Z), W) = 0$ we have the theorem.

4. Parallel Canonical Structure with Generic Submanifold of Qurternion Kaehler Manifold

Let us define the endomorphism $E : TM \to TM$, and put

(4.1) $(\overline{\nabla}_X E)\Upsilon = \nabla_X E\Upsilon - E\nabla_X \Upsilon$, For any vector field $X, \Upsilon \in TM$. The endomorphism E will be parallel if

(4.2) $\overline{\nabla}_X E = 0$ for any vector $X \in TM$

From (2.7), (2.8) and (2.14) we can obtain the following

(4.3) $$J_i \nabla_X \Upsilon + J_i h(X, \Upsilon) = J_i \overline{\nabla}_X \Upsilon$$

$= \nabla_X E\Upsilon + h(X, E\Upsilon) - A_{F\Upsilon}X + D_X F\Upsilon - G(X, \Upsilon)$

That is

(4.4) $E\nabla_X \Upsilon + F\nabla_X \Upsilon + hB(X, \Upsilon) + Ch(X, \Upsilon)$

$= \nabla_X E\Upsilon + h(X, E\Upsilon) - A_{F\Upsilon}X + D_X(F\Upsilon) - G(X, \Upsilon)$

Now compare the tangential parts, we have

$(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$ $(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$

$(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$ $(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$

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$(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$ $(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$

(4.5) $(\overline{\nabla}_X E)\Upsilon = Bh(X, \Upsilon) + A_{F\Upsilon}X + G(X, \Upsilon)^{\top}$

Therefore, for any vector fields $X, \Upsilon, Z \in TM$, we have

(4.6) $g((\overline{\nabla}_X E)\Upsilon, Z) = g(A_{F\Upsilon}Z - A_{FZ}\Upsilon + G(\Upsilon, Z)^{\top}, X)$

Now again define a new endomorphism $F : TM \rightarrow TM$, we put

(4.7) $(\overline{\nabla}_X F)\Upsilon = \nabla_X F\Upsilon - F\nabla_X \Upsilon$

Then the endomorphism will be parallel if

(4.8) $\overline{\nabla}_X F = 0$ for any vector $X \in TM$

Comparing the standard parts of (4.4), we have

(4.9) $(\overline{\nabla}_X F)\Upsilon = Ch(X, \Upsilon) - h(X, E\Upsilon) + G(X, \Upsilon)^{\perp}$

On any field of vectors $\xi \in T^{\perp}M$, it follows that

(4.10) $g((\overline{\nabla}_X F)\Upsilon, \xi) = g(Ch(X, \Upsilon) - h(X, E\Upsilon) + G(X, \Upsilon)^{\top}, \xi)$

$= -g(A_{C\xi}\Upsilon + A_{\xi}E\Upsilon - G(\Upsilon, \xi)^{\top}, X)$

Therefore we have the following

Lemma 4.1: Let M be a generic submanifold of the Kahler quaternion manifold $\overline{M}$. The E and F is Parallel, that is $\overline{\nabla}_X E = 0$ and $\overline{\nabla}_X F = 0$, if and only if

(4.11) (a $G(U,V)^{\top} = A_{FV}U - A_{FV}$ and

(b $G(X,\xi)^{\top} = A_{C\xi}X + A_{\xi}EX$

Use of any vectors $U, V, X \in TM$ and $\xi \in T^{\perp}M$.

Theorem 4.1.: In a quaternion manifold $\overline{M}$, let M be a generic submanifold of $\overline{M}$. If E and F is Parallel, then the distribution χ is integrable.

Proof: because of lemma 4.1 any vector fields $X \in \chi$ and $U \in TM$, we know $FX = 0$, which gives

(4.12) $G(U,X)^{\top} = -A_{FU}X = 0 \Rightarrow g(G(U,X),\Upsilon) = g(-A_{FU}X,\Upsilon)$

That is

(4.14) $g(G(X,\Upsilon) - J_i h(X,\Upsilon), U) = 0 \Rightarrow g(h(X,\Upsilon), FZ) = 0$

Gives that

(4.15) $g(h(X, J_i\Upsilon) - h(J_iX,\Upsilon), F_iZ) = g(2G(X,\Upsilon)^{\perp}, F_iZ)$

From lemma 3.1 this holds.
Now again from (4.9), we have

(4.16) $g((\overline{\nabla}_X F)\Upsilon,\xi) = g(Ch(X,\Upsilon) - h(X,E\Upsilon) + G(X,\Upsilon)^{\perp},\xi)$

$$= g(J_i h(X,\Upsilon),\xi) - g(h(X,E\Upsilon)\xi) + g(G(X,\Upsilon)^{\perp},\xi)$$

For any vectors $X, \Upsilon \in \chi$ and $\xi \in T^{\perp}M$. Since F is Parallel then

(4.17) $g(J_i h(X,\Upsilon),\xi) - g(h(X,E\Upsilon),\xi) + g(G(X,\Upsilon)^{\perp},\xi) = 0$

That is

(4.18) $J_i h(X,\Upsilon) = h(X, J_i\Upsilon) - G(X,\Upsilon)^{\perp}$

This implies

(4.18) $g(h(X, J_i\Upsilon) - h(J_iX,\Upsilon), F_iZ) = g(2G(X,\Upsilon)^{\perp}, F_iZ)$

So from equation (4.15) and (4.18) combined with lemma 3.1. The theorem holds.

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Data availability statement
No new data were created during the study. All data is provided in full in the results section of this paper. If any, the data used to support the findings of this study are included in the article.

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