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Asymptotic variance and MSE for $P[Y > X]$ in sampling from two one-truncation parameter families

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Abstract
In this research note we have presented some of the asymptotic theorems related to two one-truncation parameter families of distributions and use them to obtained explicit expression of asymptotic variance for U-estimable parametric function or asymptotic MSE of biased estimator. Comparison of different estimators and other inferential problem such as asymptotic minimum mean square error equivariant estimator and mean square error stabilization transformation have been tacked. Asymptotic MSE and variance of well-known stress-strength model, $P[Y > X]$ have been derived and performance of MLE and the UMVU estimator in terms of ARE and LRE have been obtained with illustrative example.
Abbreviations: UMVU; Uniformly Minimum Variance Unbiased, MLE; Maximum Likelihood Estimator, MSE; Mean Square Error, LRE; Limiting Risk Efficiency, ARE; Asymptotic Relative. _____.
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Efficiency, GUD; Generalized Uniform Distribution, pdf; probability density function, CDF; cumulative density function.

Keywords: Left-truncation, right-truncation (one-truncation) parameter family of distributions, asymptotic distributions, asymptotic variance, asymptotic mean square error, asymptotic relative efficiency, asymptotic limiting risk efficiency

Introduction

One-truncation parameter family of distributions mean that left or right extremity of range of distribution involve parameter. Rohatgi (1989) [14] consider probability density function (pdf) of two independent right-truncation distributions as

$$f_1(x;\theta_1) = q_1(\theta_1)h_1(x);\ a < x < \theta_1 \quad \ldots 1.1/1$$
$$f_2(y;\theta_2) = q_2(\theta_2)h_2(y);\ a < y < \theta_2 \quad \ldots 1.2/2$$

Cumulative distribution function (CDF) as

$$F_1(u;\theta_1) = \frac{q_1(\theta_1)}{q_1(u)} \quad \ldots 1.3/3$$
$$F_2(u;\theta_2) = \frac{q_2(\theta_2)}{q_2(u)} \quad \ldots 1.4/4$$

With

$$f_{11} = f_1(\theta_1;\theta_1) = q_1(\theta_1)h_1(\theta_1) = -q_1^{-1}(\theta_1)\frac{\partial}{\partial\theta_1}q_1(\theta_1) \quad \ldots 1.5/5$$
$$f_{22} = f_2(\theta_2;\theta_2) = q_2(\theta_2)h_2(\theta_2) = -q_2^{-1}(\theta_2)\frac{\partial}{\partial\theta_2}q_2(\theta_2) \quad \ldots 1.6/6$$

And (pdf) of two independent left-truncation distributions as

$$f_3(z;\theta_3) = q_3(\theta_3)h_3(z);\ \theta_3 < z < b \quad \ldots 1.7/7$$
$$f_4(w;\theta_4) = q_4(\theta_4)h_4(w);\ \theta_4 < w < b \quad \ldots 1.8/8$$

CDF as

$$F_3(v;\theta_3) = 1 - \frac{q_3(\theta_3)}{q_3(v)} \quad \ldots 1.9/9$$

$$F_4(v;\theta_4) = 1 - \frac{q_4(\theta_4)}{q_4(v)} \qquad \ldots 1.10/10$$

With

$$f_{33} = f_3(\theta_3;\theta_3) = q_3(\theta_3)h_3(\theta_3) = q_3^{-1}(\theta_3)\frac{\partial}{\partial\theta_3}q_3(\theta_3) \qquad \ldots 1.11/11$$

$$f_{44} = f_4(\theta_4;\theta_4) = q_4(\theta_4)h_4(\theta_4) = q_4^{-1}(\theta_4)\frac{\partial}{\partial\theta_4}q_4(\theta_4) \qquad \ldots 1.12/12$$

Respectively, where $-\infty < a < b < \infty$ are known constants, q_i; $(1,2,3,4)$ are everywhere differentiable functions, h_i; $(1,2,3,4)$ are absolutely continuous functions for two sample problem and obtained explicit expression for the UMVU estimator of any U-estimable parametric function $g = g(\theta_i,\theta_j)$; $i \neq j = 1,2,3,4$ by virtue of explicit expression for the Uniformly Minimum Variance Unbiased (UMVU) estimator of reliability functional, $P[Y > X]$ have obtained.
Amongst many others Beg and Singh (1979) [2], Beg (1980a, 1980b, 1980c, 1983) [3, 4, 5, 6], Dixit, Ali and Woo (2003) [9], Dixit and Phal (2009) [10], obtained the UMVU estimator of $P[Y > X]$ when the independent random variables $X$and Y follows truncation parameter families. References to previous work and other technical aspect of this problem [see Simonoff *et al.* (1986) [16], Samuel *et al*. (2002)] [12].
Problem of obtaining explicit expression of variance or MSE was not properly or optimally addressed due to intractability of exact distributions to be needed. The application of asymptotic result are therefore of interest which provide a novel method to address the above challenges by asymptotic distribution of any differentiable function of complete sufficient statistic for $\theta = (\theta_i,\theta_j)$; $i \neq j = 1,2,3,4$ at reasonable order derived in section 2. Special case for single one (left/right)-truncation distribution is discussed as corollary.
Performance of Maximum Likelihood Estimator (MLE) and the UMVU estimator (traditional competitors) in terms of Limiting Risk Efficiency (LRE), Asymptotic Relative (ARE) have been compared with illustrative example in section 3. Asymptotic results regarding inferential problems such as asymptotic minimum mean square error equivariant estimator and Mean Square Error (MSE) stabilization transformation have been obtained in section 4 and 5. Section 6 devoted to asymptotic MSE of MLE and asymptotic variance of the MUVU estimator of reliability functional $P[Y > X]$ to compare performance relatively in terms of LRE and ARE and this have been illustrated example.

2. Asymptotic distributions [main results]

Let $X_1, X_2, \ldots X_n$ and $Z_1, Z_2, \ldots, Z_n$ be random samples with common pdf f_1 and f_3 where f_1 and f_3 defined in (1.1/1) and (1.7/7) respectively. We assume that $X_i's$ and $Z_i's$ are independent. Further θ_1 and θ_3 are not functionally related. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ and $Z_{1:n} < Z_{2:n} < \cdots < Z_{n:n}$ be corresponding sets of order statistics. The complete sufficient statistics of parameters $\theta = (\theta_1,\theta_3)$ is $X = (X_{n:n}, Z_{1:n})$ and it's joint pdf at $x = (x_1, z_1)$ is given by

$$f_X(x) = n^2\left[\frac{q_1(x_1)q_3(z_1)}{q_1(\theta_1)q_3(\theta_3)}\right]^{-n} q_1(x_1)q_3(z_1)h_1(x_1)h_3(z_1)dx_1dz_1,$$
$$a < x_1 < \theta_1, \theta_3 < z_1 < b \qquad \ldots 2.1/13$$

After expanding $q_1(x_1)$ and $q_3(z_1)$ appear in bracket of right hand side of $2.1/13$ by Taylor series expansion around θ_1 and θ_3 one gets

$$f_X(x) = n^2\left[1 + (x_1-\theta_1)\frac{\frac{d}{d\theta_1}q_1(\theta_1)}{q_1(\theta_1)} + \frac{\sum_{k=2}^{\infty}\left(\frac{r^k}{k!}\right)(x_1-\theta_1)^r\frac{d^r}{d^r\theta_1}q_1(\theta_1)}{q_1(\theta_1)}\right]^{-n}$$
$$.\left[1 + (z_1-\theta_3)\frac{\frac{d}{d\theta_3}q_3(\theta_3)}{q_3(\theta_3)} + \frac{\sum_{k=2}^{\infty}\left(\frac{r^k}{k!}\right)(z_1-\theta_3)^r\frac{d^r}{d^r\theta_3}q_3(\theta_3)}{q_3(\theta_3)}\right]^{-n}$$
$$.\, q_1(x_1)q_3(z_1)h_1(x_1)h_3(z_1)dx_1dz_1 \qquad \ldots 2.2/14$$

By using probability integral transformation
$S_{r:n} = F_1(x_{r:n};\theta_1)$ and $V_{r:n} = F_3(z_{r:n};\theta_3)$

The asymptotic distribution of
$A_n = (\theta_1 - X_{n:n})$ And $C_n = (Z_{1:n} - \theta_3)$

Can be obtained. It is easy to show that
$K_n = n(1 - S_{n:n})$ and $K_1 = nV_{1:n}$

Are asymptotically independent and converge in distribution to exponential random variables (See, David, 1981. P.267). Thus for joint asymptotic distribution of X given in 2.2/14 we have following lemma.

Lemma 2.1 In the above setup

$$U_1 = nf_{11}[\theta_1 - X_{n:n}] = nf_{11}A_n \xrightarrow{d} E[0,1] \qquad \ldots 2.3/15$$

And

$$U_3 = nf_{33}[Z_{1:n} - \theta_3] = nf_{33}C_n \xrightarrow{d} E[0,1] \qquad \ldots 2.4/16$$

Are independent exponential random variables with mean zero and variance one and at the order $o(n^{-1})$ joint asymptotic pdf of $X = (X_{n:n}, Z_{1:n})$ is
$f_U(u) = EXP[-u_1 - u_3]$

Where
$U = [U_1, U_3]$, $u = [u_1, u_3]$ and
$T \xrightarrow{d} E[\mu, \sigma]$, denotes that T follows exponential distribution with it's pdf as
$E[\mu, \sigma] = \sigma^{-1} E[\sigma^{-1}(t - \mu)]; \mu < t < \infty, \sigma > 0.$
A similar argument leads to the following lemmas.

Lemma 2.2: Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ and $Y_{1:n} < Y_{2:n} < \ldots < Y_{n:n}$ be corresponding set of order statistic of random samples $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots Y_n$ from f_i; $i = 1, 2$ then the largest order statistics $X = (X_{n:n}, Y_{n:n})$ is complete sufficient statistics of $\theta = (\theta_1, \theta_2)$. Then at the order $o(n^{-1})$ random variables $U_1 \xrightarrow{d} E[0,1]$ and

$$U_2 = nf_{22}[\theta_2 - Y_{n:n}] = nf_{22}B_n \xrightarrow{d} E[0,1] \quad \ldots 2.5/17$$

Where
$B_n = (\theta_2 - Y_{n:n}).$

Lemma 2.3 Let $Z_{1:n} < Z_{2:n} < \ldots < Z_{n:n}$ and $W_{1:n} < W_{2:n} < \ldots < W_{n:n}$ be corresponding set of order statistic of random samples $Z_1, Z_2, \ldots Z_n$ and $W_1, W_2, \ldots W_n$ from f_i; $i = 3, 4$ then the smallest order statistics $Z = (Z_{1:n}, W_{1:n})$ is complete sufficient statistics of $\theta = (\theta_3, \theta_4)$. Then at the order $o(n^{-1})$ random variables $U_3 \xrightarrow{d} E[0,1]$ and

$$U_4 = nf_{44}[W_{1:n} - \theta_4] = nf_{44}D_n \xrightarrow{d} E[0,1] \quad \ldots 2.6/18$$

Where
$D_n = (W_{1:n} - \theta_4).$

Theorem 2.1 In the set-up of lemma 2.2, let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ and $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$ be corresponding sets of order statics of random samples $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots, Y_n$ from f_i;$(i = 1, 2)$ respectively. Let $g = g(\theta)$, $\theta = (\theta_1, \theta_2)$ be a real valued differentiable parametric function with non-zero first order partial derivative with respect to θ_1 and θ_2. Then at the order $o(n^{-1})$ asymptotic distribution of random variable.

$$[g(X) - g(\theta)] = \lambda_1 U_1 + \lambda_2 U_2 \xrightarrow{d} P[0, \lambda] \quad \ldots 2.7/19$$

Where
$X = (X_{n:n}, Y_{n:n}),$
$\lambda = (\lambda_1, \lambda_2),$

$$\lambda_1 = \lambda_1(g) = -\frac{g_1}{nf_{11}} \quad \ldots 2.8/20$$

$$\lambda_2 = \lambda_2(g) = -\frac{g_2}{nf_{22}} \quad \ldots 2.9/21$$

$$g_i = \frac{\partial}{\partial \theta_i} g; i = 1, 2$$

And $P[\xi, \eta\lambda]$ denoted by general gamma distribution [see Johson and Kotz, 1970, Vol.-2, P 222] with pdf as

$$f_U(u; \xi, \eta\lambda) = \frac{\lambda_1 Exp\left[-\frac{u-\xi}{\eta\lambda_1}\right]}{\lambda_1 - \lambda_2} + \frac{\lambda_2 Exp\left[-\frac{u-\xi}{\eta\lambda_2}\right]}{\lambda_2 - \lambda_1} \quad \ldots 2.10/22$$

$u > \xi > 0, \eta > 0, \lambda_1 \neq \lambda_2.$

Proof: Consider Young's form of Taylor series expansion of $g(X)$ around θ such as

$$g(X) = g(\theta) + [X_{n:n} - \theta_1]g_1 + [Y_{n:n} - \theta_2]g_2$$
$$+ \sum_{k=2}^{\infty} \left(\frac{r^k}{k!}\right) \left[\alpha_1 \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_2}\right]^k g(\theta)$$
$$\alpha_1 = [X_{n:n} - \theta_1], \alpha_2 = [Y_{n:n} - \theta_2]$$

And simplifying we get

$$[g(X) - g(\theta)] = [-f_{11}g_1]nf_{11}[\theta_1 - X_{n:n}]$$
$$+ [-f_{22}g_2]nf_{22}[\theta_2 - Y_{n:n}] \sum_{k=2}^{\infty} \left(\frac{r^k}{k!}\right) \left[\alpha_1 \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_2}\right]^k g(\theta).$$

Using $U_i; i = 1, 2$ defined in 2.3/15 and 2.5/17 and substituting $\lambda_i; i = 1, 2$ defined in 2.8/20 and 2.9/21 the $[g(X) - g(\theta)]$ will be

$$[g(X) - g(\theta)] = \sum_{i=1}^{2} \lambda_i U_i \xrightarrow{d} P[0, \lambda]$$

And from 2.10/22, theorem 2.1 follow.
A similar argument leads to the following theorems.

Theorem 2.2: In the set-up of lemma 2.3 Let $Z_{1:n} < Z_{2:n} < \cdots < Z_{n:n}$ and $W_{1:n} < W_{2:n} < \cdots < W_{n:n}$ be corresponding sets of order statistics of random samples $Z_1, Z_2, \ldots Z_n$ and $W_1, W_2, \ldots, W_n$ from $f_i; (i = 3, 4)$ respectively. Let $g = g(\theta)$, $\theta = (\theta_3, \theta_4)$ be a real valued differentiable parametric function with non-zero first order partial derivative with respect to θ_3 and θ_4. Then at the order $o(n^{-1})$ asymptotic distribution of random variable.

$$[g(Z) - g(\theta)] = \lambda_3 U_3 + \lambda_4 U_4 \xrightarrow{d} P[0, \lambda] \qquad \ldots 2.11/23$$

Where
$Z = (Z_{1:n}, W_{1:n})$,
$\lambda = (\lambda_3, \lambda_4)$,
$$\lambda_3 = \lambda_3(g) = \frac{g_3}{nf_{33}} \qquad \ldots 2.12/24$$
$$\lambda_4 = \lambda_4(g) = \frac{g_4}{nf_{44}} \qquad \ldots 2.13/25$$
$g_i = \frac{\partial}{\partial \theta_i} g; i = 3, 4$

And $P[\xi, \eta\lambda]$ denoted by general gamma distribution defined in 2.10/22.

Theorem 2.3: In the set-up of lemma 2.1 Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ and $Z_{1:n} < Z_{2:n} < \cdots < Z_{n:n}$ be corresponding sets of order statistics of random samples $X_1, X_2, \ldots X_n$ and $Z_1, Z_2, \ldots, Z_n$ from $f_i; (i = 1, 3)$ respectively. Let $g = g(\theta)$, $\theta = (\theta_1, \theta_3)$ be a real valued differentiable parametric function with non-zero first order partial derivative with respect to θ_1 and θ_3. Then at the order $o(n^{-1})$ asymptotic distribution of random variable
$$[g(X) - g(\theta)] = \lambda_1 U_1 + \lambda_3 U_3 \xrightarrow{d} P[0, \lambda] \qquad \ldots 2.14/26$$

Where
$X = (X_{n:n}, Z_{1:n})$,
$\lambda = (\lambda_1, \lambda_3)$
$g_i = \frac{\partial}{\partial \theta_i} g; i = 1, 3.$
And $P[\xi, \eta\lambda]$ denoted by general gamma distribution defined in 2.10/22.

Corollary 2.1: In setup of theorem 3.3, suppose $g(\theta_1, \theta_3) = k_1(\theta_1)$ function of θ_1 only and $g(\theta_1, \theta_3) = k_2(\theta_3)$ function of θ_3 only then $\tilde{k}_1(\theta_1) = k_1(X_{n:n})$ and $\tilde{k}_2(\theta_3) = k_2(Z_{1:n})$ are MLE of $k_1(\theta_1)$ and $k_2(\theta_3)$ whereas the UMVU estimator of $k_1(\theta_1)$ and $k_2(\theta_3)$ obtained by Tate (1959) are
$$\hat{k}_1(\theta_1) = k_1(X_{n:n}) + [f_1(X_{n:n}; X_{n:n})]^{-1} \frac{\partial}{\partial X_{n:n}} k_1(X_{n:n})$$
And
$$\hat{k}_2(\theta_3) = k_2(Z_{1:n}) - [f_3(Z_{1:n}; Z_{1:n})]^{-1} \frac{\partial}{\partial Z_{1:n}} k_2(Z_{1:n}).$$

Tacking $g(\theta_1, \theta_3) = k_1(\theta_1)$ and $g(\theta_1, \theta_3) = k_2(\theta_3)$ in theorem 2.3 following asymptotic distributions for one sample [right/left] truncation parameter family of distributions reduced from general gamma distribution defined in 2.10/22 as special cases.
i) $\left[\tilde{k}_1(\theta_1) - k_1(\theta_1)\right] \to E[0, n^{-1}\Delta_1]$; if $k_1'(\theta_1) > 0$
ii) $\left[\hat{k}_1(\theta_1) - k_1(\theta_1)\right] \to E[-n^{-1}\Delta_1, n^{-1}\Delta_1]$; if $k_1'(\theta_1) > 0$
iii) $\left[\tilde{k}_1(\theta_1) - k_1(\theta_1)\right] \to E[0, -n^{-1}\Delta_1]$; if $k_1'(\theta_1) < 0$,
iv) $\left[\hat{k}_1(\theta_1) - k_1(\theta_1)\right] \to E[n^{-1}\Delta_1, -n^{-1}\Delta_1]$; if $k_1'(\theta_1) < 0$

And
i) $\left[\tilde{k}_2(\theta_3) - k_2(\theta_3)\right] \to E[0, n^{-1}\Delta_2]$; if $k_2'(\theta_3) > 0$
ii) $\left[\hat{k}_2(\theta_3) - k_2(\theta_3)\right] \to E[-n^{-1}\Delta_2, n^{-1}\Delta_2]$; if $k_2'(\theta_3) > 0$
iii) $\left[\tilde{k}_2(\theta_3) - k_2(\theta_3)\right] \to E[0, -n^{-1}\Delta_2]$; if $k_2'(\theta_3) < 0$
iv) $\left[\hat{k}_2(\theta_3) - k_2(\theta_3)\right] \to E[n^{-1}\Delta_2, -n^{-1}\Delta_2]$; if $k_2'(\theta_3) < 0$

Where
$$\Delta_i = [f_i(\theta_i; \theta_i)]^{-1} g_i'(\theta_i) = [f_i(\theta_i; \theta_i)]^{-1} \frac{\partial}{\partial \theta_i} g(\theta_i); i = 1, 3..$$

ARE and LRE of MLE and the UMVU Estimator
In the set-up of theorem 2.1, theorem 2.2 and theorem 2.3, Rohatgi (1989) [14] obtained the UMVU estimator $\hat{g} = \hat{g}(\theta)$ for any real valued differentiable (with non-zero first order partial derivative) U-estimable parametric function $g = g(\theta)$; $[\theta = (\theta_1, \theta_2), \theta = (\theta_3, \theta_4)\ and\ \theta = (\theta_1, \theta_3)]$ as
$$\hat{g} = \hat{g}(\theta_1, \theta_2) = g(X_{n:n}, Y_{n:n}) + \frac{\frac{\partial}{\partial X_{n:n}} g(X_{n:n}, Y_{n:n})}{nf_1(X_{n:n}, X_{n:n})} + \frac{\frac{\partial}{\partial Y_{n:n}} g(X_{n:n}, Y_{n:n})}{nf_2(Y_{n:n}, Y_{n:n})}$$

$$+\frac{\frac{\partial^2}{\partial X_{n:n}\partial Y_{n:n}}g(X_{n:n},Y_{n:n})}{n^2 f_1(X_{n:n},X_{n:n}) f_2(Y_{n:n},Y_{n:n})},$$

$$\hat{g}=\hat{g}(\theta_3,\theta_4)=g(Z_{1:n},W_{1:n})-\frac{\frac{\partial}{\partial Z_{1:n}}(Z_{1:n},W_{1:n})}{n f_3(Z_{1:n},Z_{1:n})}-\frac{\frac{\partial}{\partial W_{1:n}}(Z_{1:n},W_{1:n})}{n f_4(W_{1:n},W_{1:n})}$$

$$+\frac{\frac{\partial^2}{\partial Z_{1:n}\partial W_{1:n}}(Z_{1:n},W_{1:n})}{n^2 f_3(Z_{1:n},Z_{1:n}) f_4(W_{1:n},W_{1:n})}$$

And

$$\hat{g}=\hat{g}(\theta_1,\theta_3)=g(X_{n:n},Z_{1:n})+\frac{\frac{\partial}{\partial X_{n:n}}g(X_{n:n},Z_{1:n})}{n f_1(X_{n:n},X_{n:n})}-\frac{\frac{\partial}{\partial Z_{1:n}}g(X_{n:n},Z_{1:n})}{n f_3(Z_{1:n},Z_{1:n})}$$

$$-\frac{\frac{\partial^2}{\partial X_{n:n}\partial Z_{1:n}}g(X_{n:n},Z_{1:n})}{n^2 f_1(X_{n:n},X_{n:n}) f_3(Z_{1:n},Z_{1:n})}$$

Respectively, respectively.
In order to compare the performance of MLE and the UMVU estimator, the following theorems have been established.

Theorem 3.1: In the set-up of theorem 2.1, if $\tilde{g}=\tilde{g}(\theta)=\tilde{g}(\theta_1,\theta_2)=g(X_{n:n},Y_{n:n})$ is MLE estimator and $\hat{g}=\hat{g}(\theta)=\hat{g}(\theta_1,\theta_2)$ is the UMVU estimator of U-estimable parametric function $g=g(\theta)=g(\theta_1,\theta_2)$ then at the order $o(n^{-3})$ the LRE of $\hat{g}$ relative to $\tilde{g}$ is given by

$$LRE[\hat{g},\tilde{g}]=2\left[1+\frac{\alpha\beta}{\alpha^2+\beta^2}\right] \qquad \ldots 3.1/27$$

Particularly

$$LRE[\hat{g},\tilde{g}]=\begin{cases} 3; if\ \alpha=\beta \\ 2; if\ \alpha\beta=0 \\ 1; if\ \alpha=-\beta \\ \in(2,4); if\ \alpha\beta>0 \\ \in(0,2); if\ \alpha\beta<0 \end{cases} \qquad \ldots 3.2/28$$

Where

$$\alpha=\frac{g_1}{n f_{11}} \qquad \ldots 3.3/29$$

And

$$\beta=\frac{g_2}{n f_{22}} \qquad \ldots 3.4/30$$

Proof: Using theorem 2.1 at the order $o(n^{-1})$ the asymptotic distribution of $[\tilde{g}-g]$ is
$[\tilde{g}-g]\to P[0,-\alpha,-\beta]$,

With bias $B[.]$ as
$B[\tilde{g}]=-\alpha-\beta$,

At the order $o(n^{-2})$ and MSE as
$MSE[\tilde{g}]=2(\alpha^2+\beta^2+\alpha\beta)$,

At the order $o(n^{-3})$.
Taking $G(X_{n:n},Y_{n:n})=\hat{g}(\theta_1,\theta_2)$ derived by Rohatgi (1989) [14],

We get
$[G(X_{n:n},Y_{n:n})-G(\theta_1,\theta_2)]=\lambda_1(G)U_1+\lambda_2(G)U_2$.

Which reduces to
$U=[\hat{g}(\theta_2,\theta_4)-g(\theta_2,\theta_4)]=\lambda_1 U_1+\lambda_2 U_2$

And at the order $o(n^{-1})$ and the asymptotic distribution of $[\hat{g}-g]$ is
$[\hat{g}-g]\to P[\alpha+\beta,-\alpha,-\beta]$

With variance $V[.]$ as
$V[\hat{g}]=\alpha^2+\beta^2$

At the order $o(n^{-3})$. Straight forward computation leads to the theorem 3.1.
Analogues argument leads to the following theorems.

Theorem 3.2 In the set-up of theorem 2.2, if $\tilde{g} = \tilde{g}(\theta) = \tilde{g}(\theta_3, \theta_4) = g(Z_{1:n}, W_{1:n})$ is MLE and $\hat{g} = \hat{g}(\theta) = \hat{g}(\theta_3, \theta_4)$ is the UMVU estimator of U-estimable parametric function $g = g(\theta) = g(\theta_3, \theta_4)$ respectively then $[\tilde{g} - g] \to P[0, \gamma, \zeta]$ and $[\hat{g} - g] \to P[-\gamma - \zeta, \gamma, \zeta]$ at the order $o(n^{-1})$ and at the order $o(n^{-3})$ the LRE of $\hat{g}$ relative to $\tilde{g}$ is given by

$$LRE[\hat{g}, \tilde{g}] = 2\left[1 + \frac{\gamma\zeta}{\gamma^2 + \zeta^2}\right] \quad \ldots 3.5/31$$

Where

$$\gamma = \frac{g_3}{nf_{33}} \quad \ldots 3.6/32$$

$$\zeta = \frac{g_4}{nf_{44}} \quad \ldots 3.7/33$$

Theorem 3.3: In the set-up of theorem 2.3, if $\tilde{g} = \tilde{g}(\theta) = \tilde{g}(\theta_1, \theta_3) = g(X_{n:n}, Z_{1:n})$ is MLE and $\hat{g} = \hat{g}(\theta) = \hat{g}(\theta_1, \theta_3)$ is the UMVU estimator of U-estimable parametric function $g = g(\theta) = g(\theta_1, \theta_3)$ respectively then $[\tilde{g} - g] \to P[0, -\alpha, \gamma]$ and $[\hat{g} - g] \to P[\alpha - \gamma, -\alpha, \gamma]$ at the order $o(n^{-1})$ and at the order $o(n^{-3})$ the LRE of $\hat{g}$ relative to $\tilde{g}$ is given by

$$LRE[\hat{g}, \tilde{g}] = 2\left[1 - \frac{\alpha\gamma}{\alpha^2 + \gamma^2}\right] \quad \ldots 3.8/34$$

AREs of the UMVU estimator with respect MLE does not exist.
Example 3.1 Consider uniform distribution on interval $(0, \theta_1)$ and right truncated exponential distribution with pdf as $f_1(x; \theta_1) = \theta_1^{-1}; 0 < x < \theta_1$

And

$$f_2(x; \theta_2) = \frac{e^{-y}}{1-e^{-\theta_2}}; 0 < y < \theta_2$$

Respectively. Let $g = g(\theta_1, \theta_2) = e^{-\theta_1\theta_2}$ then using theorem 3.1, one can obtain following results

$$[\tilde{g} - g] \to P\left[0, \frac{g\theta_1\theta_2}{n}, \frac{g\theta_1(1 - e^{-\theta_2})}{ne^{-\theta_2}}\right]$$

$$B[\tilde{g}] = \frac{g\theta_1[(\theta_2 - 1)e^{-\theta_2} + 1]}{ne^{-\theta_2}}$$

$$MSE[\tilde{g}] = 2\left(\frac{g^2\theta_1^2}{n^2e^{-2\theta_2}}\right)\left[\left[\theta_2^2e^{-2\theta_2} + \left(1 - e^{-\theta_2}\right)^2\right] + \theta_2e^{-\theta_2}\left(1 - e^{-\theta_2}\right)\right]$$

$$[\hat{g} - g] \to P\left[-\frac{g\theta_1[(\theta_2 - 1)e^{-\theta_2} + 1]}{ne^{-\theta_2}}, \frac{g\theta_1\theta_2}{n}, \frac{g\theta_1(1 - e^{-\theta_2})}{ne^{-\theta_2}}\right]$$

$$V[\hat{g}] = \frac{g^2\theta_1^2\left[\theta_2^2e^{-2\theta_2} + \left(1 - e^{-\theta_2}\right)^2\right]}{n^2e^{-2\theta_2}}$$

And

$$LRE[\hat{g}, \tilde{g}] = 2\left[1 + \frac{\theta_2e^{-\theta_2}(1 - e^{-\theta_2})}{\theta_2^2e^{-2\theta_2} + \left(1 - e^{-\theta_2}\right)^2}\right].$$

Minimum mean square error equivariant estimator

Let $\mathfrak{x}$ be sample space for two samples, one sample taken from f_1 defined 1.1/1 and the other sample taken from f_2 defined 1.7/7 and let

$$\mathfrak{F} = \left\{\mathrm{P}_{(\theta_1, \theta_2)}: (\theta_1, \theta_2) \in \Theta\right\}$$

Be family of probability measure on $(\mathfrak{x}, \mathfrak{B})$, where $\mathfrak{B}$ is σ field. Let G be group of transformation on $\mathfrak{x}$ such that $g\mathfrak{x} = \mathfrak{x}$, $g\mathfrak{B} = \mathfrak{B}$, for all $g \in \mathrm{G}$,

$$\mathrm{G} = \begin{bmatrix} g: g(X_k, Y_l) = g(X_k, Y_l) + g(c_1, c_2), for\ all\ k = 1, 2, \ldots, n, \\ l = 1, 2, \ldots, n, -\infty < c_1 < \infty, -\infty < c_2 < \infty. \end{bmatrix}.$$

Let G^* be induced group of transformation on parametric space, Θ defined as

$$P_{(\theta_1, \theta_2)}[B] = P_{g^*(\theta_1, \theta_2)}[g^*B]$$

Where $B \in \mathfrak{B}$ and $g^* \in G^*$. Assume that G^* is one to one and $g^*\Theta = \Theta$ for all $g^* \in G^*$, such that
$G^* = [g^*: g^*(\theta_1, \theta_2) = g(\theta_1, \theta_2) + g(c_1, c_2), -\infty < c_1 < \infty, -\infty < c_2 < \infty]$.

Further the square error loss function be invariant with respect to G^*. Let induced group of transformation on space of complete sufficient statistics (T_1, T_2)
$\bar{G} = [\bar{g}: \bar{g}(\theta_1, \theta_2) = g(T_1, T_2) + g(c_1, c_2), -\infty < c_1 < \infty, -\infty < c_2 < \infty]$.

Theorem 4.1 In the set-up of theorem 2.1 and with respect to above transformation G at the order $o(n^{-2})$ the asymptotic minimum MSE equivariant estimator, $\bar{g}(\theta_1, \theta_2)$ of $g(\theta_1, \theta_2)$ is

$$\bar{g}(\theta_1, \theta_2) = g(X_{n:n}, Y_{n:n}) + \frac{\frac{\partial}{\partial X_{n:n}}g(X_{n:n}, Y_{n:n})}{nf_1(X_{n:n}; X_{n:n})} + \frac{\frac{\partial}{\partial Y_{n:n}}g(X_{n:n}, Y_{n:n})}{nf_2(Y_{n:n}; Y_{n:n})} \quad \ldots 4.1/29$$

Proof: Using theorem 2.1 at the order $o(n^{-1})$ asymptotic distribution of $[g(X_{n:n}, Y_{n:n}) + g(c_1, c_2) - g(\theta_1, \theta_2)]$ is

$$[g(X_{n:n}, Y_{n:n}) + g(c_1, c_2) - g(\theta_1, \theta_2)] \to P[g(c_1, c_2), -\alpha, -\beta] \quad \ldots 4.2/30$$

And α and β are same as defined in 3.3/25 and 3.4/26. Further at the order $o(n^{-3})$ the asymptotic MSE is

$$MSE[g(X_{n:n}, Y_{n:n}) + g(c_1, c_2) - g(\theta_1, \theta_3)] = \begin{bmatrix} 2\eta^2[\alpha^2 + \beta^2 + \alpha\beta] \\ \\ + 2(\alpha + \beta)g(c_1, c_2) + g^2(c_1, c_2) \end{bmatrix} \quad \ldots 4.3/31$$

Minimization of 4.3/31 at the order $o(n^{-2})$ gives

$$g(c_1, c_2) = \frac{\frac{\partial}{\partial\theta_1} g(\theta_1, \theta_2)}{n f_1(\theta_1; \theta_1)} + \frac{\frac{\partial}{\partial\theta_2} g(\theta_1, \theta_2)}{n f_2(\theta_2; \theta_2)} \quad \ldots 4.4/32$$

Replacing $X_{n:n}$ for θ_1 and $Y_{n:n}$ for θ_2 in 4.4/32, the theorem follows.
A Similar argument leads to the following theorem 4.2 and theorem 4.3.

Theorem 4.2: In the set-up of theorem 2.2 and with respect to above transformation G at the order $o(n^{-2})$ the asymptotic minimum MSE equivariant estimator, $\bar{g}(\theta_3, \theta_4)$ of $g(\theta_3, \theta_4)$ is

$$\bar{g}(\theta_3, \theta_4) = g(X_{1:n}, Y_{1:n}) - \frac{\frac{\partial}{\partial X_{1:n}} g(X_{1:n}, Y_{1:n})}{n f_3(X_{1:n}; X_{1:n})} - \frac{\frac{\partial}{\partial Y_{1:n}} g(X_{1:n}, Y_{1:n})}{n f_4(Y_{1:n}; Y_{1:n})}.$$

Theorem 4.3: In the set-up of theorem 2.3 and with respect to above transformation G at the order $o(n^{-2})$ the asymptotic minimum MSE equivariant estimator, $\bar{g}(\theta_1, \theta_3)$ of $g(\theta_1, \theta_3)$ is

$$\bar{g}(\theta_1, \theta_3) = g(X_{n:n}, Y_{1:n}) + \frac{\frac{\partial}{\partial X_{n:n}} g(X_{n:n}, Y_{1:n})}{n f_1(X_{n:n}; X_{n:n})} - \frac{\frac{\partial}{\partial Y_{1:n}} g(X_{n:n}, Y_{1:n})}{n f_3(Y_{1:n}; Y_{1:n})}.$$

5. MSE stabilized transformation

The asymptotic inference regarding the parameters θ_1 and θ_2 require the asymptotic distributions of $(\theta_1 - X_{n:n})$ and $(\theta_2 - Y_{n:n})$ respectively but as MSE depends on parameters and the purpose will not be served until suitable MSE stabilization transformation is applied. The following theorem serve the purpose of suitable MSE stabilization transformation.

Theorem 4.1: In the set-up of theorem 3.1 if $g = g(\theta_1, \theta_2)$ is MSE stabilized transformation then
$g = g(\theta_1, \theta_2) = \pm \log q_1(\theta_1) \pm \log q_2(\theta_2)$.

Proof After obtaining solution of

$$\frac{\partial}{\partial\theta_1} \log q_1(\theta_1) = \pm f_1(\theta_1; \theta_1)$$

And

$$\frac{\partial}{\partial\theta_2} \log q_2(\theta_2) = \pm f_2(\theta_2; \theta_2)$$

And using theorem 3.1, theorem 4.1 follow. In the set-up of theorem 3.2 and theorem 3.3 it is easy to obtain MSE stabilized transformation by analogues arguments.

6. Asymptotic MSE and variance $P[Y > X]$

Statistical inference concerning the reliability functional; $P[Y > X]$ has been of great interest to researchers who are doing research in reliability theory, survival analysis, engineering and industrial statistics. Asymptotic MSE of MLE and asymptotic variance of the UMVU estimator of reliability functional has been obtained and compared their performance in terms of LRE. In the set-up of theorem 2.1, $\tilde{g} = \tilde{g}(\theta_1, \theta_2) = g(X_{n:n}, Y_{n:n})$ is MLE of

$$\begin{aligned} g = g(\theta_1, \theta_2) = P_{(\theta_1, \theta_2)}[Y > X] &= \int_a^{\theta_1} P_{(\theta_2)}[Y > x]\, f_1(x; \theta_1) dx \\ &= \int_a^{Min(\theta_1, \theta_2)} P_{(\theta_2)}[Y > x]\, f_1(x; \theta_1) dx \\ &= \begin{cases} \int_a^{\theta_2} f_1(x; \theta_1)\, dx \int_x^{\theta_2} f_2(y; \theta_2)\, dy; \theta_1 > \theta_2 \\ \\ \int_a^{\theta_1} f_1(x; \theta_1)\, dx \int_x^{\theta_2} f_2(y; \theta_2)\, dy; \theta_1 < \theta_2. \end{cases} \end{aligned}$$

And the UMVU estimator of g derived by Rohatgi (1989) [14] is

$$\hat{g} = \hat{g}(\theta_1, \theta_2) = \hat{P}_{(\theta_1, \theta_2)}[Y > X] = \begin{cases} (n-1)\left[\frac{(n-1)g + F_{12}}{n^2}\right]; \theta_1 > \theta_2 \\ \\ 1 + (n-1)\left[\frac{(n-1)(g-1) - F_{21}}{n^2}\right]; \theta_1 < \theta_2 \end{cases}$$

After replacing $X_{n:n}$ for θ_1 and $Y_{n:n}$ for θ_2, where F_{12} and F_{21} are cdf of f_1 at θ_2 and f_2 at θ_1.

Using theorem 3.1 asymptotic distribution of $[\tilde{g}-g]$ is

$$[\tilde{g}-g] \rightarrow \begin{cases} P\left[0, \frac{g}{n}, \frac{g-F_{12}}{n}\right]; \theta_1 > \theta_2 \\ P\left[0, \frac{g+F_{21}-1}{n}, \frac{g-1}{n}\right]; \theta_1 < \theta_2, \end{cases}$$

At the order $o(n^{-1})$ with bias, $B[.]$ as

$$B[\tilde{g}] = \begin{cases} \frac{2g-F_{12}}{n}; \theta_1 > \theta_2 \\ \frac{2g+F_{21}-1}{n}; \theta_1 < \theta_2, \end{cases}$$

at the order $o(n^{-2})$ and MSE as

$$MSE[\tilde{g}] = \begin{cases} 2\left[\frac{3g^2+F_{12}^2-3gF_{12}}{n^2}\right]; \theta_1 > \theta_2 \\ 2\left[\frac{3(g-1)^2+F_{21}^2-2(g-1)F_{21}}{n^2}\right]; \theta_1 < \theta_2 \end{cases}$$

At the order $o(n^{-3})$ and at the order $o(n^{-1})$ the asymptotic distribution of $[\hat{g}-g]$ is

$$[\hat{g}-g] \rightarrow \begin{cases} P\left[-\frac{2g-F_{12}}{n}, \frac{g}{n}, \frac{g-F_{12}}{n}\right]; \theta_1 > \theta_2 \\ P\left[-\frac{2g+F_{21}-1}{n}, \frac{g+F_{21}-1}{n}, \frac{g-1}{n}\right]; \theta_1 < \theta_2 \end{cases}$$

With asymptotic variance as

$$V[\hat{g}] = \begin{cases} \frac{2g^2+F_{12}^2-2gF_{12}}{n^2}; \theta_1 > \theta_2 \\ \frac{2(g-1)^2+F_{21}^2-2(g-1)F_{21}}{n^2}; \theta_1 < \theta_2 \end{cases}$$

At the order $o(n^{-1})$. Hence at the order $o(n^{-3})$ asymptotic LRE of $\hat{g}$ relative to $\tilde{g}$ is

$$LRE[\hat{g},\tilde{g}] = \begin{cases} 2\left[1+\frac{g^2-gF_{12}}{2g^2+F_{12}^2-2gF_{12}}\right]; \theta_1 > \theta_2 \\ 2\left[1+\frac{(g-1)^2+(g-1)F_{21}}{2(g-1)^2+F_{21}^2-2(g-1)F_{21}}\right]; \theta_1 < \theta_2. \end{cases}$$

Plant develops into the reproductive phase of growth, a mat of smaller roots grows near the surface to a depth of approximately 1/6-th of maximum depth achieve [See G. Ooms and K.L. Moore (1991) [13]]. Dixit (2003 and 2009) [9, 10] assumed that a set of random variables $X_1, X_2, \ldots, X_n$ represents the masses of roots where out of n random variables some of these roots (say n) have different masses therefore, those masses have different uniform distribution with unknown parameters and these k observations are distributed with Generalized Uniform Distribution (GUD).

Example 5.1: Consider generalized uniform distribution on interval $(0,\theta_1)$ and on interval $(0,\theta_2)$ with pdf as

$$f_1(x;\theta_1) = \frac{\rho+1}{\theta_1^{\rho+1}}x^{\rho}; < x < \theta_1$$

and

$$f_2(y;\theta_2) = \frac{\tau+1}{\theta_2^{\tau+1}-\theta_3^{\tau+1}}x^{\rho}; 0 < y < \theta_2.$$

If $g = P_{(\theta_1,\theta_2)}[Y > X]$ then g will be

$$g = \begin{cases} \left[\frac{\tau+1}{\rho+\tau+2}\right]\left(\frac{\theta_2}{\theta_1}\right)^{\rho+1}; \theta_1 > \theta_2 \\ 1-\left[\frac{\rho+1}{\rho+\tau+2}\right]\left(\frac{\theta_1}{\theta_2}\right)^{\tau+1}; \theta_1 < \theta_2. \end{cases}$$

Using the results of section 6 following results are easy to obtain.

$$[\tilde{g}-g] \to \begin{cases} P\left[0, \frac{1}{n}\left[\frac{\tau+1}{\rho+\tau+2}\right]\left(\frac{\theta_2}{\theta_1}\right)^{\rho+1}, -\frac{1}{n}\left[\frac{\rho+1}{\rho+\tau+2}\right]\left(\frac{\theta_2}{\theta_1}\right)^{\rho+1}\right]; \ \theta_1 > \theta_2 \\ P\left[0, \frac{1}{n}\left[\frac{\tau+1}{\rho+\tau+2}\right]\left(\frac{\theta_1}{\theta_2}\right)^{\tau+1}, -\frac{1}{n}\left[\frac{\rho+1}{\rho+\tau+2}\right]\left(\frac{\theta_1}{\theta_2}\right)^{\tau+1}\right]; \ \theta_1 < \theta_2, \end{cases}$$

$$B[\tilde{g}] = \begin{cases} \frac{\tau-\rho}{n(\rho+\tau+2)}\left(\frac{\theta_2}{\theta_1}\right)^{\rho+1}; \ \theta_1 > \theta_2 \\ \frac{\tau-\rho}{n(\rho+\tau+2)}\left(\frac{\theta_1}{\theta_2}\right)^{\tau+1}; \ \theta_1 < \theta_2, \end{cases}$$

$$MSE[\tilde{g}] = \begin{cases} 2\left[\frac{(\tau+1)^2+(\rho+1)^2-(\tau+1)(\rho+1)}{n^2(\rho+\tau+2)^2}\right]\left(\frac{\theta_2}{\theta_1}\right)^{2\rho+2}; \ \theta_1 > \theta_2 \\ 2\left[\frac{(\tau+1)^2+(\rho+1)^2-(\tau+1)(\rho+1)}{n^2(\rho+\tau+2)^2}\right]\left(\frac{\theta_1}{\theta_2}\right)^{2\tau+2}; \ \theta_1 < \theta_2, \end{cases}$$

$$V[\hat{g}] = \begin{cases} \left[\frac{(\tau+1)^2+(\rho+1)^2}{n^2(\rho+\tau+2)^2}\right]\left(\frac{\theta_2}{\theta_1}\right)^{2\rho+2}; \ \theta_1 > \theta_2 \\ \left[\frac{(\tau+1)^2+(\rho+1)^2}{n^2(\rho+\tau+2)^2}\right]\left(\frac{\theta_1}{\theta_2}\right)^{2\tau+2}; \ \theta_1 < \theta_2 \end{cases}$$

And

$$LRE[\hat{g},\tilde{g}] = \begin{cases} 2\left[1 - \frac{(\tau+1)(\rho+1)}{(\tau+1)^2+(\rho+1)^2}\right]; \ \theta_1 > \theta_2 \\ 2\left[1 - \frac{(\tau+1)(\rho+1)}{(\tau+1)^2+(\rho+1)^2}\right]; \ \theta_1 < \theta_2. \end{cases}$$

It is interesting note following:
1. In both cases, $\theta_1 > \theta_2$ or $\theta_1 < \theta_2$ the $LRE[\hat{g},\tilde{g}]$ remain same.
2. If $\tau = 0$, that is f_2 is uniform distribution on interval $(0,\theta_2)$ then in both cases $\theta_1 > \theta_2$ or $\theta_1 < \theta_2$ the $LRE[\hat{g},\tilde{g}]$ remain same and is $2\left[1 - \frac{\rho+1}{1+(\rho+1)^2}\right]$.
3. If $\tau = \rho = 0$, that is f_1 and f_2 are uniform distribution on interval $(0,\theta_1)$ and $(0,\theta_2)$ then in both cases $\theta_1 > \theta_2$ or $\theta_1 < \theta_2$ the $LRE[\hat{g},\tilde{g}]$ remain same and is 1, because $\alpha = -\beta$ which agreed to 3.2/18. In this case asymptotically $\hat{g}$ is not more efficient then $\tilde{g}$.

References

1. Beg and Singh. Beg (1980a, 1980b, 1980c, 1983), Dixit, Ali and Woo (2003), Dixit and Phal (2009) 1979.
2. Beg MA, Singh N. Estimation of $P[Y < X]$ for two Pareto distribution, IEEE Transaction on Reliability 1979;28(5):411-414.
3. Beg MA. Estimation of $P[Y < X]$ for truncation parameters distributions, Communication in Statistics Theory and methods 1980a;9:327-345.
4. Beg MA. On the Estimation of $P[Y < X]$ for two parameter exponential distribution, Metrika 1980b;27:29-34.
5. Beg MA. Estimation of $P[Y < X]$ for exponential family, Transactions on Reliability 1980c;29:158-159.
6. Beg MA. Unbiased estimators for truncation and scale parameters, American Journal of Mathematical and Management Sciences 1983;3:251-274.
7. Beg MA, Singh N. Estimation of $P[Y < X]$ for two Pareto distribution, IEEE Transaction on Reliability 1979;28(5):411-414.
8. David H. Order Statistics, Wiley, New York 1969.
9. Dixit UJ, Masoom A, Woo J. Efficient estimation of parameters of a uniform distribution in the presence of outliers, IEEE Transaction on Reliability 2003;29(4):363-369.
10. Dixit UJ, Phal KD. Estimation of $P[Y < X]$ for the uniform distribution in the Presence of Outliers, Journal of Probability and Statistical Science 2009;7(1):11-18.
11. Johnson NL, Kotz S, Pensky M. Distributions in Statistics Continuous Univariate Distributions-1, John Wiley & Sons Inc., 1970.
12. Kotz Samuel, Yan L, Pensky M. The Stress-Strength Model and its Generalizations Theory and Applications, World Scientific, London 2002.
13. Ooms G, Moore KL. A model assay for genetic and environmental changes in the architecture of intact roots systems of plants grown *in vitro*, Plant Cell, Tissue and Organ Culture 1991;27:129-139.
14. Rohatgi VK. Unbiased estimation of parametric function in sampling from two one-truncation parameter families, Australian Journal of Statistics 1989;31(2):327-332.
15. Serfling RF. Approximation theorems of Mathematical Statistics, John Wiley, New York 1980.
16. Simonoff JS, Hochberg Y, Reiser B. Alternative estimation procedures for $P[X < Y]$ in categorized data, Biometrics 1986;42:895-907.
17. Tate RF. Unbiased estimation; function of location and scale parameters, Annals of Mathematical Statistics 1959;30:341-366.