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Etebong P Clement
Department of Statistics,
University of Uyo, P. M. B. Uyo,
Nigeria

Uko S Jim
Department of Mathematics,
University of Uyo, P. M. B. Uyo,
Nigeria

Corresponding Author:
Etebong P Clement
Department of Statistics,
University of Uyo, P. M. B. Uyo,
Nigeria

Statistical mechanics in economics: An application of Brownian motion in modeling prices of assets

Etebong P Clement and Uko S Jim

Abstract
The Black Option Pricing Model in recent times has been under strong criticism. The Black Models assume that the probability of extreme price changes is negligible, when in reality stock prices in the stock market are subject to fluctuations. Again, the modern formulation of statistical models is based on the description of the physical system by an ensemble that represents all possible configurations of the system and the probability of realizing each configuration. Therefore, the probability of extreme price changes should not be neglected. Consequently, this work set out to develop a model (the Modified Geometric Brownian Motion Model (MGBMM)) for perpetual warrant option for prices of assets (shares of stock) traded in a perfect market with an arbitrary stock price in the warrant. Reasons that support the Modified Geometric Brownian Motion Model as an appropriate model in a perfect market are presented with a completely specified strategy for managing option investment which permits practical testing of the model's efficacy. Thus, illustrating how the notion of Statistical Mechanics is applied in economics to model the prices of assets in the financial market.

Keywords: Black option pricing model, Brownian motion, modified geometric Brownian motion model, perfect market, perpetual warrant option and statistical mechanics

1. Introduction

Certain Special classes of Stochastic Process have undergone extensive mathematical development. The Brownian Motion Process is the most renowned and historically the first which was thoroughly investigated [1].

As a physical phenomenon the Brownian motion was discovered by the English Botanist R. Brown in 1827. A mathematical description of this phenomenon was first derived from the laws of physics by Einstein in 1905 [2]. Since then the subject has made considerable progress. The physical theory was further perfected by Smoluchowiski, Fokker, Planck, Burger, Furth, Ornstein, Uhlenback, Chandrasekhar, Kramers, and other [1, 3]. The mathematical theory was slower in developing because the exact mathematical description of the model posed difficulties, whereas some of the questions to which the physicists sought answers on the basis of model were quite simple and intuitive [3]. Many of the answers were obtained in a heuristic way by Bachelier in his 1900 dissertation [4] whereas the first concise mathematical formulation of the theory was given by Wiener in 1918 [5].

Statistical Mechanics is a branch of physics that applies probability theory, which contains mathematical tools for dealing with large populations, to the study of the thermodynamic behavior of physical systems composed of a large number of particles [6].

Statistical Mechanics was initiated in 1870 with the work of Austrain Physicist Ludwig Boltzmann, much of which was collectively published in Boltzmann's 1896 lectures on Gas Theory [7] in the proceedings of the Vienna academy and other societies. The term Statistical Mechanics was first proposed by J. William Gibbs in 1902 [8]. According to Gibbs, the term 'statistical' in the context of Mechanics (i.e. statistical mechanics) was first used by the Scottish Physicist James Clerk Maxwell in 1871 who also presented the first-ever statistical law in Physics [9]. The fundamental postulate in Statistical mechanics (also known as the equal a priori probability postulate) is:

Given an isolated system in equilibrium, it is found with equal probability in each of its accessible microstates. This postulate is a fundamental assumption in Statistical Mechanics. It states that a system in equilibrium does not have any preference for any of its available microstates. The modern formulation of Statistical Mechanics is based on the description of the physical system by an ensemble that represents all possible configuration of the system and the probability of realizing each configuration [10].

Brownian motion is the seemingly random movement of particles suspended in a fluid or the mathematical model used to describe such random movement, often called the Particle Theory [11]. The mathematical model of Brownian motion has several real world applications. An often quoted example is Stock fluctuation. However, movements in share prices may arise due to unforeseen events which do not repeat themselves, and physical and economic phenomenon are not comparable [12].

The first person to describe the mathematics behind Brownian Motions was Thorvald N, Thiele in 1880 in a paper on the method of least squares. This was followed independently by Lonis Bachelier in 1900 in his Ph.D thesis" The Theory of Speculation". In which he presented a stochastic analysis of the stock and option market [3]. However, it was Albert Einstein and Mariam Smoluchowiski who independently brought the solution of the problem to the attention of physicist, and presented it as a way to indirectly confirm the existence of atoms and molecules [11].

In mathematics, Brownian motion is described by the Wiener process; a continuous-time stochastic process named in honour of Norbert Wiener. It is one of the best known Levy process (stochastic process with stationary independent increment) and frequently occurs in Pure and Applied Mathematics, Economics and Physics [2]. A Geometric Brownian Motion (GBM) (also Exponential Brownian Motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian

Motion [13], also called a Wiener process. It is applicable to mathematical modeling of some phenomena in financial markets. It is used particularly in the field of option pricing because a quantity that follows Geometric Brownian motion may take any positive value, and only the fractional changes of the random variate are significant. This is a reasonable approximation of stock price dynamics except for rare events [13].

A large part of any science is the ability to create testable hypotheses based on a fundamental understanding of the objects of study and prove or contradict the hypotheses through repeatable studies [14]. In this light, statistic is a language for representing theories and provides tools for testing their validity. For example, in the theory of option pricing due to Black- Scholes and Merton, a model for movement of stock prices is posited, and in conjunction with basis theory which states that a riskless investment will receive the risk free rate of return, the researchers reason that a value can be assign to an option that is independent of the expected future value of the stock [15].

The Black option pricing model is under strong criticism: [16] argued that the Black model merely recast existing widely used models in terms of practically impossible "dynamics hedging" rather than 'risk' in order to make them more compatible with mainstream neoclassical economic theory. Added they opined that "Reliance on models based on incorrect axioms has clear and large effects; while [17] argued that Black models assume that the probability of extreme price changes is negligible, when in reality, stock prices are much Jerkier than this". Works in this aspect include [18-20].

Consequently, this paper develops a model (the Modified Geometric Brownian Motion Model) to evaluate the worth of perpetual warrant stock option in a stock market system and provides a completely specified strategy for managing option investment which permits practical testing of the model's efficacy.

2. Main Results

A Stochastic Process is a sequence of random variables [21]. It could be a discrete parameter process or of a continuous parameter process. The Theory of Statistical Processes is concerned with the investigation of the structure of families of random variables $X(t)$, where t is a parameter running over a suitable index set T [22]. A very important example of a Stochastic Process is the Brownian motion process.

Brownian motion is a stochastic process $\{X(t); t \geq 0\}$ with the following properties:

a. Every increment $X(t+s) - X(s)$ is normally distributed with mean zero and variance σ_t^2; σ is a fixed parameter.

b. For every pair of disjoint time interval $[t_1, t_2]$, $[t_3, t_4]$ say $t_1 < t_2 \leq t_3 < t_4$, the increment $X(t_4) - X(t_3)$ and $X(t_2) - X(t_1)$ are independently random variables with distribution given in (a), and similarly for n disjoint time intervals where n is an arbitrary positive integer.

c. $X(0) = 0$ and $X(t)$ is continuous at $t = 0$.

Thus, a displacement $X(t+s) - X(s)$ is independent of the past, or alternatively, if $X(s) = x_0$ is known, then no further knowledge of the values of $X(\beta)$ for $\beta < s$ has any effect on our knowledge of the probability law governing $X(t+s) - X(s)$. This means that if $t > t_0 > t_1 > t_2 > \cdots > t_n$ the

$$\Pr[X(t) \leq x | X(t_0) = x_0;\ X(t_1) = x_1, \dots, X(t_n) = x_n] = \Pr[X(t) \leq x | X(t_0) = x_0] \quad 2.1$$

This is a statement of the Markov Character of the process. The process X

$\overline{\overline{X}}(t) = X(t)|\sigma$ is a Brownian Motion Process having a variance parameter of one, called Standard Brownian Motion.

Following from property

(a) With $\sigma^2 = 1$, we hav

$$\Pr[X(t) \leq x | X(t_0)] = \Pr[\, X(t) - X(t_0) \leq x - x_0] = \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{\infty} \exp\left[\frac{\alpha^2}{2(t-t_0)}\right] d\alpha \quad 2.2$$

The consistency of property (b) with property (a) follows from well-known properties of the normal distributions for instance, if

$t_1 \leq t_2 \leq t_3$; the $X(t_3) - X(t_1) = [X(t_3) - X(t_2)] + [X(t_2) - X(t_1)]$ 2.3

Following from property (a) the density function

$$f(x_1, \ldots x_n) = \Pr(x_1, t_1) \Pr(x_2 - x_1, t_2 - t_1), \ldots, Pr(x_n - x_{n-1}, t_n - t_{n-1}) \quad 2.4$$

Where

$$\Pr(x, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right) \quad 2.5$$

Indeed one can compute in principle any set of conditional probabilities desired using equation (2.4).
According to the Markov property, it is known that, if $t_1 \leq t_2 \leq t_3$; then

$$f\{X(t_3)|X(t_1), X(t_2) = f\{X(t_3)|X(t_2)\} \quad 2.6$$

However, the density of $X(t_2)|X(t_1), X(t_3)$is also of interest. For definiteness, supposes $X(t_1) - X(t_3) = 0$ and say specifically that $t_1 = 0, t_3 = 0$ and $t_2 = t(0 < t < 1)$. By equation (2.4) the joint density of $X(t)$and $X(1)$ is:

$$f(x, y) = \frac{1}{\sqrt{2\pi t(1-t)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{t} + \frac{(y-x)^2}{1-t}\right)\right] \quad 2.7$$

$$f_t(x|X(0) = X(1) = 0) = \frac{1}{\sqrt{2\pi t(1-t)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{2t(1-t)}\right)\right] - \infty < x < \infty \quad 2.8$$

Let$\{X(t), t \geq 0\}$ be a Brownian motion process with drift and diffusion co-efficient σ^2. The process defined by

$$y(t) = e^{X(t)}, t \geq 0 \quad 2.9$$

is called Geometric Brownian Motion. The State space S is the interval $(0, \infty)$.

Since $y(t) = y(0)e^{X(t)-X(0))}$

Then

$$E[\, y(t)|y(0) = y] = yE[e^{X(t)-X(0)}\,] = y\exp\{t[\mu + \frac{1}{2}\sigma^2\,] = ye^{t\alpha} \quad 2.10$$

Where

$$\alpha = \mu + \frac{1}{2}\sigma^2 \text{ and } E[y^2(t)|y(0) = y] = y^2E[e^{2[X(t)-X(0)]}]ye^{t\alpha}$$
$$= y^2\exp\{t[2\mu + \frac{1}{2}4\sigma^2\,]$$
$$= y^2\exp\{2t[\mu + \sigma^2\,] \quad 2.11$$

And variance $Var[y(t)|y(0) = y] = E[y^2(t)|y(0) = y] - [E[\, y(t)|y(0) = y]]^2$
$$= y^2\exp\{2t[\mu + \sigma^2\,] - y^2e^{2t\alpha}$$
$$= y^2[e^{2t\alpha}(e^{t\sigma^2} - 1)] \quad 2.12$$

Equations (2.10), (2.11) and (2.12) are derived following the principle of characteristic function of the normal distribution.

Theorem 2.1: Let $\{X(t); t \geq 0\}$ be a Brownian Motion Process with drift $\mu \neq 1$ and variance σ^2 and suppose $X(0) = x$. The probability that the process reaches the level $b > x$ before hitting $a < x$ is given by:

$$\Pr[X(t) = b|X(0)] = \frac{\exp(-2\mu x|\sigma^2) - \exp(-2\mu a|\sigma^2)}{\exp(-2\mu b|\sigma^2) - \exp(-2\mu a|\sigma^2)} \quad 2.13$$

Corollary 2.1: Let $X(t)$ be a Brownian Motion process with drift $\mu < 0$. Let

$$W = \max_{0 \leq t < \infty}(X(t) - X(0)) \quad 2.14$$

Thus Whas the exponential distribution

$$\Pr[W \geq w] = e^{-\lambda w} \quad 2.15$$

Where $\lambda = 2|\mu|/\sigma^2$

Let $z > 0$ be fixed and $T = T_z$ be the first, if any, the process reaches the level z.

$$T = T_z = \begin{cases} inf\ \{t: X(t) \geq z\ if\ X(t) \geq z\ for\ some\ t \geq 0 \\ \infty\ if\ X(t) < z\ \forall t \geq 0 \end{cases} \quad 2.16$$

Set $\theta = \lambda\mu + \frac{1}{2}\lambda^2\sigma^2$. Then $\mathrm{Var}(t) = \exp\{\lambda X(t) - \theta t\}$ is a martingale, and if $X(0) = 0$, $E[V(T \wedge t)] = 1$

$$E[\exp\{\lambda X(T \wedge t) - \theta(T \wedge t)\}] = 1 \quad 2.17$$

Let $\lambda \geq 0$ be sufficiently large to ensure $\theta \geq 0$. Then $0 \leq (T \wedge t) \leq e^{\lambda z}$

Lemma 2.1: Let W be an arbitrary random variable satisfying $E\{|W|] < \infty$ and let T be a Marker Time for which $\Pr\{T < \infty\} = 1$. Then

$$\lim_{n\to\infty} E\{W\, I_{T>n}] = 0 \quad 2.18$$

And

$$\lim_{n\to\infty} E\{W\, I_{T>n}] = E[W] \quad 2.19$$

Using Lemma 2.1

$$\lim_{t\to\infty} V(T \wedge t) = \begin{cases} exp\ [\lambda z - \theta T \geq z\ if\ T < \infty \\ 0\ if\ T = \infty \end{cases} \quad 2.20$$

so that $E[V(T \wedge t)] = 1$

$$e^{\lambda z} E\left[e^{-\theta T}\right] = 1$$

$$E\left[e^{\theta T}\right] = e^{\lambda z} \quad 2.21$$

Relating λ and θ we have

$$\lambda\mu + \frac{1}{2}\lambda^2\sigma^2 - \theta = 0 \quad 2.22$$

or

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma\theta}}{\sigma^2} \quad 2.23$$

we require $\lambda \geq 0$

$$\lambda = \frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma\theta} - \mu\right) \quad 2.24$$

When $\mu < 0, T$ has a definite probability, so that, T is infinite with positive probability, and

$$\Pr\{T < \infty\} = \lim_{\theta\to 0} E\left[e^{-\theta T}\right] = \lim_{\theta\to 0} \exp\left[-\frac{z}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma\theta} - \mu\right)\right] = \exp\left(-\frac{2z|\mu|}{\sigma^2}\right) \quad 2.25$$

Equation (2.25) conforms to corollary 2.1 when $0;\mu \geq 0, T < \infty$ with certainty, and the Laplace transform is:

$$E\left[e^{\theta T}\right] = \exp\left[-\frac{z}{\sigma^2}\left(\sqrt{\mu^2 + 2\sigma\theta} - \mu\right)\right] \quad 2.26$$

Theorem 2.2: Let $X(t)$ be a Brownian motion with drift $\mu \geq 0$. Let $z > X(0) = x$ be given and let $T_z$be the first time the process reaches the level Z. Conditioned on $X(0) = x$, T_z has the probability density function

$$f(t, x, z) = \frac{z - x}{\sigma\sqrt{2\pi t^3}} \exp\left[\frac{(z - x - \mu t)^2}{\sigma^2 t}\right] \quad 2.27$$

3. Model Simulations

Let β_0 be the current stock price. Let β_t be a future stock price. Let β^* be the ratio of the future stock price to the stock price. Then

$$\beta^* = \frac{\beta_0}{\beta_t} \quad 3.1$$

If $\beta^*$is anticipated or predicted as being favourable. Then:

1. A number of buyers are anticipated to participate in the stock market
2. The demand of the anticipated buyers would tend to raise the current stock price β_0.
3. Similarly, if β^* Is anticipated or predicted as being unfavorable. Then:
4. A number of sellers are anticipated to participate in the stock market
5. Their participation would tend to depress the current stock market price β_0.
   However, if is predicted as being neither favourable nor unfavorable (that is when the price ratios β^*over non-overlapping time intervals are independent), the stock market system will stabilized. According to [23], the stock market system will converge in accordance with the Central Theorem for Markov Chain Convergence (CTMCC).

Definition 3.1: A warrant is an option to buy a fixed number of shares in a given stock at a stated stock price at any time during a specified time period. The profit to the stock holder for such an option is the excess of the stock market price over the option price.

3.1 Model Assumptions

The following assumptions hold for the model:

1. The stock holder(s) would purchase shares at the stated price and resell them on the market price and thus realize the potential profit,
2. The stock price reaches some specified levelφ when the warrant is exercised the first time,
3. The stated stock price in the warrant is, λ so that the potential profit upon exercising the warrant option at a stock market price of φ' is $\varphi - \lambda$ by an appropriate choice of units,
4. Only $\varphi > \lambda$ is considered, since one would not purchase at the stated stock market price of λ if the current stock market price were lower (that is $\lambda > \varphi$ '),
5. The warrant option is a perpetual warrant option. In owning such an option, one is foregoing, in part at least, direct ownership of the stock with the rate of increase of the warrant option as:

$$\alpha = \mu + \frac{1}{2}\sigma^2 \quad 3.2$$

Per-unit time, since according to equation (2.10). $E[\, y(t) | y(0) = y] = y\exp\{t[\mu + \frac{1}{2}\sigma^2\,]$

One requires a higher rate of return, $\theta > \alpha$ from the option or equivalently, discounts the potential profit if $\varphi - \lambda$ at a rate $-\theta$ per unit time. Let $T(\alpha)$ be the first time the stock price reaches the level φ. Then, the discounted potential profit (f) to the warrant option holder is:

$$e^{-\theta T(\alpha)}\left[Y\left(Y(\alpha)\right) - \lambda\right] = e^{-\theta T(\alpha)}(\varphi - \lambda) \quad 3.3$$

3.2 Simulation Algorithms

The following steps are involved in the simulation process:

1. Compute the expected discounted profit (f).
2. Choose φ reasonably, to maximize the expected discounts profit (f)
3. Compute the probability density function for $T(\alpha)$ according to Theorem 2.2 and equation (2.26)
4. Find the profit maximizing level if $\varphi = \varphi^*$
5. In terms of the Brownian motion $T(\alpha)$is the first time that $T(\alpha)X(t) = \ln Y(t)$ 3.4 reaching the level $\ln\varphi$ using equation (2.26) with:
6. $Z = \ln\varphi$
7. $x = \ln y$

We have,

$$[e^{-\theta T(\alpha)} | Y(0) = y] = \left(\frac{y}{\varphi}\right)^p \quad 3.5$$

Where

$$p = \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\theta}{\sigma^2}} - \frac{\mu}{\sigma^2} \quad 3.6$$

Letting $g(y,\varphi)$ be the expected discounted profit (f), we have

$$g(y,\varphi) = (\varphi - \lambda)\left[e^{-\theta T(\alpha)} \middle| Y(0) = y\right] \quad 3.7$$

Following from equation (3.5) we have

$$g(y,\varphi) = (\varphi - \lambda)\left(\frac{y}{\varphi}\right)^p \quad 3.8$$

Maximizing profit level φ^* To find the profit maximizing level $\varphi = \varphi^*$, we differentiate equation (3.8) with respect to φ and equate it to zero

$$\frac{dg}{d\varphi}\Big|_{\varphi=\varphi^*} = -p(\varphi - \lambda)\left(\frac{y}{\varphi}\right)^{p+1}.\frac{1}{y} + \left(\frac{y}{\varphi}\right)^{p} = \left(\frac{y}{\varphi}\right)^{p+1}\left\{\frac{-p(\varphi-\lambda)}{y} + \frac{y}{\varphi}\right\} \frac{dg}{d\varphi}\Big|_{\varphi=\varphi^*} = 0, \left(\frac{y}{\varphi}\right)^{p+1}\left\{\frac{-p(\varphi-\lambda)}{y} + \frac{y}{\varphi}\right\} = 0$$

$$\varphi^* = \frac{p\lambda}{p-1} \quad 3.9$$

Hence

$$g(y, \varphi^*) = (\varphi^* - \lambda)\left(\frac{y}{\varphi^*}\right)^{p} \quad 3.10$$

Accordingly

$$g(y, \varphi^*) = \frac{p\lambda}{p-1}\left(\frac{y(p-1)}{p\lambda}\right)^{p} \quad 3.11$$

4. Result and Discussion

Geometric Brownian motion is often used to model prices of assets, say shares of stock, that are traded in a perfect market, such prices are nonnegative and usually exhibit oscillatory behavior comprised of exponential growth intermitted with exponential decay over the long-run, two properties possessed by Geometric Brownian Motion. More independently, if $t_0 < t_1 < \cdots < t_n$ are time points the successive ratios $Y(t_1)/Y(t_0), \ldots, Y(t_n)/Y(t_0)$ are independently random variables, so that, the percentage changes over non-overlapping time intervals are independent. We have presented reasoning that supports the Geometric Brownian Motion as an appropriate model in a perfect market. We have also given examples in which GBM model is used to evaluate the worth of a perpetual warrant in a stock. We consider only perpetual warrants, options having no expiration dates. A reasonable strategy has been adopted by exercising the warrant option the first time the stock price reaches some specified level. The Black option pricing model in recent time has been under strong criticisms. According to (Bouchaud, 2008), the Black models assume that the probability of extreme price changes is negligible, when in reality stock prices in the stock market are subject to fluctuations. Again the modern formulation of statistical models is based on the description of the physical system by an ensemble that represents all possible configurations of the system and the probability of realizing each configuration. Therefore, the probability of extreme price changes should not be neglected. Consequently, this work set out to design a model (the Modified Geometric Brownian Motion Model) for the perpetual warrant option for prices of assets (shares of stock) traded in a perfect market with an arbitrary stock price in the warrant. Reasons that support the Modified Geometric Brownian Motion model as an appropriate model in a perfect market are present. Thus, illustrating how the notion of statistical mechanics is applied in economics to model the prices of assets in the financial markets, with a completely specified strategy for managing option investment which permits practical testing of the efficacy of the model

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